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P. Draminsky

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by

P. Draminsky

dr. techn.

Burmeister & Wain, Copenhagen

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Mechanical Engineering Series

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Summary

A theory of non-linear crankshaft vibrations is developed, fully considering the effects of impulse distortion and mass variation. A case of "secondary resonance" in a 10-cylinder engine is described and calculated. Representing the non-linearities as "fictive forces" acting on a linear system a great simplification of detail calculations is obtained.

Contents

Notations used in the paper

α crank angle, measured from the "main position" where the total moment of inertia is minimum.

$\beta_c = \dfrac{\frac{1}{2}g_o R^2}{m_c}$ coefficient of oscillating mass pr. cylinder.

$\beta_{pc} = \beta_c(-\lambda,\ 1,\ \lambda)$ coefficient of mass variation pr. cyl. of order p = (1, 2, 3).

$\beta_p = \dfrac{\Sigma(\beta_{pc} m_c \Delta^2)}{\Sigma m \Delta^2}$ coefficient of total mass variation in multi-cylinder engines.

β short for β_2

$\beta_{ab} = \dfrac{\Sigma(\beta_c m_c \Delta_a \Delta_b)}{\Sigma m \Delta_b^2}$ coefficient of coupled mass variation (of 2. order) from vibration form A to B.

γ phase angle of impulse at main position.

φ - - - vibration at - -

θ angle between cranks.

Δ relative vibration amplitudes in a vibration form.

$\rho = \dfrac{\Sigma r \Delta^2}{\omega_e \Sigma m \Delta^2}$ dimensionless coefficient of total damping.

λ = R/L connecting rod relation.

x vibration amplitude, radian.

a, b, c maximum values of vibration amplitudes at the point where $\Delta = 1$.

a_s or $a_{ps} = \dfrac{\Sigma(K_p \Delta)}{\omega_e^2 \Sigma m \Delta^2}$ reduced or relative impulse of order p, also called "static amplitude", radian.

a_u or $a_{pu} = \dfrac{\Sigma(K_p \Delta^2)}{\omega_e^2 \Sigma m \Delta^2}$ reduced impulse for subharmonic excitation.

a^*_{pu} same, with oscillating-mass impulse included in K_p.

a'_{q-p}, b'_q etc. fictive force or artificial impulse of order indicated and produced by vibration denoted by the letter. Measured relatively as a_s and a_u.

$F = \dfrac{a}{a_s}$ magnification factor.

J total, equivalent inertia moment, variable.

$\Sigma m \Delta^2$ - - - - , mean value.

K_p cylinder impulse of order p pr. revolution.

q number of vibrations pr. revolution.

ω_o angular velocity of crankshaft.

ω phase velocity of vibration.

ω_e - - - free vibrations, calculated for constant masses.

$\xi \quad = \frac{\omega}{\omega_e} = \frac{q\omega_o}{\omega_e}$ relative resonance.

1. Introduction

In certain types of 10-cylinder two-stroke engines torsional vibrations of 8. order have been observed, which could not be explained by the common calculation methods, as the geometric resultant of the 10 impulses of 8. order for the adopted crank position was very small and should not be able to exite vibrations of a size more than about one fifth of the actually measured.

Luckily, these cases were not of serious consequences as the vibrations were found somewhat above normal revolutions of the engine and would therefore not occur during normal service, but as such an unexpected phenomenon might, in other cases, arise at a more inconvenient range, a full explanation of it was wanted in order to avoid it.

Thorough investigations showed, that there could be no considerable action of impulses from the propeller, coupled vibrations from sources outside the engine, etc. Furthermore, the stress curve about the resonance point was of a shape similar to ordinary resonance curves as if the impulse of 8. order were about 5 times bigger than it really was, and this artificial, extra impulse must be sought within the vibrating crankshaft system itself.

Hereafter it was clear, that the vibrations must be a kind of "non-linear" vibrations which cannot be calculated by the common, "linear" vibration theory. Some knowledge of non-linear vibrations is thus necessary for those responsible for the calculation of torsional vibrations in a great variety of big and costly engines. Even if non-linear vibrations can be expected only in a few and special cases, the study of them is in any case interesting and will give an increased feeling of safety also in the normal calculations.

The appellation "linear" is here used for systems determined by linear differential equations with constant coefficients, and "non-linear" for systems with variable coefficients, whether the variations are time functions or functions of the vibration amplitude x. In the mathematical terminology only the latter systems are truly

"non-linear" (namely non-linear in x), but in vibration theory there is not much difference between the two kinds of variation because x is in itself a periodic time function when there are steady vibrations, and it is therefore more important to have handy names to distinguish between systems with constant coefficients and with variable coefficients. Yet, when we have to distinguish between the two kinds of variation (the time-depending and the vibration-depending) we shall here call the former systems "freely variable" and the latter "true non-linear systems", while the cases with both time- and vibration-depending variations (to which the crankshaft system belongs) will sometimes be called "guided variable systems". These names have a distinct physical meaning, and the "freedom" or "guidance" of the variations will always appear directly from the constructive features of the system.

The problems concerning a "second order theory" of torsional vibrations were some 30 years ago thoroughly treated by Trefftz, Kluge, Grammel and others[1], and the subject may have been considered as exhausted with these highly theoretical papers. But while their fundamentals, chiefly based on application of Lagrange's equations, are of unchanging validity, their subsequent treatment of technical details and approximations seem to-day unsatisfactory and insufficient from the engineer's point of view. On the other hand, there is to-day an extensive litterature on non-linear vibrations, but neither is this sufficient for the use of the mechanical engineer meeting new and unexpected problems in his practice. Especially there is very little to be found about non-linear vibrations in multi-mass systems, although dangerous vibrations in practice always concern multi-mass systems. It has so far been left to the practical man to find out how he shall best substitute his real, multi-mass system with an equivalent one-mass system which the mathematician will calculate for him with any wanted accuracy and subtlety.

The investigation of the above mentioned unexpected vibrations resulted in a rather complete theory of non-linear crankshaft vibrations and easy and yet in first approximation completely correct calculation methods as described in the following. Besides, it was

1) see references at end of paper.

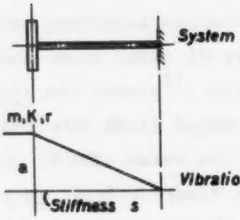
necessary to develop certain basic principles for a general theory of non-linear vibrations in multi-mass systems. These matters are mostly found in sections 4 and 8-10.

The parts printed with closer line distance contain (besides detail calculations for the control of energy balances etc.) the theory of subharmonic vibrations, while the main text goes straightly for the theory of "secondary resonance" which seems to be the more important matter, in any case for crankshaft vibrations. A general theory of non-linear vibrations must comprise both phenomena but also clearly distinguish between them.

2. Linear vibrations in one-mass and multi-mass systems

It will be appropriate first to recapitulate the calculation of steady vibrations in linear systems with constant coefficients, and to introduce a special terminology which is very convenient, not to say necessary, for treating the intricate problems of non-linearity.

In a one-mass fixed system fig. 1, consisting of a swing



$$\omega_e^2 = \frac{s}{m} \qquad \xi = \frac{\omega}{\omega_e}$$

$$a_s = \frac{K}{m\omega_e^2} \qquad \rho = \frac{r}{m\omega_e}$$

$$a = a_s \cdot F \qquad F = \frac{1}{\sqrt{(1-\xi^2)^2 - \rho^2\xi^2}}$$

Fig. 1. Torsional vibrations in a linear one-mass system.

mass with inertia moment m kgcmsec^2 and a shaft with stiffness s kgcm/rad, acted upon by an impulse $K \sin(\omega t + \gamma)$ kgcm and a damping $r \frac{dx}{dt}$ kgcm, the angular vibration amplitude x is determined by the differential equation

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + sx = K \sin(\omega t + \gamma) \tag{1}$$

Dividing with $m\omega_e^2 = s$ we get:

$$\frac{d^2x}{\omega_e^2 dt^2} + \rho \frac{dx}{\omega_e dt} + x = a_s \sin(\omega t + \gamma) \tag{2}$$

where $a_s = \frac{K}{s}$ means a reduced or relative impulse of dimension vibration amplitude (here dimensionless), also to be called "static

amplitude" as it is the deflection which the system would get for a static force K, and $\rho = \frac{r}{m\omega_e}$ is a dimensionless damping coefficient.

Introducing $\xi = \frac{\omega}{\omega_e}$, the relative frequency of the impulse in relation to the frequency of free vibrations, we get

$$\xi^2 \frac{d^2x}{d(\omega t)^2} + \rho\,\xi \frac{dx}{d(\omega t)} + x = a_s \sin(\omega t + \gamma) \tag{3}$$

The continuos solution (disregarding tuning-in) is

$$\left.\begin{aligned}
&x = a \sin(\omega t + \varphi) \qquad \text{where} \qquad a = a_s \cdot F \\
&\text{the magnification factor} \qquad F = \frac{1}{\sqrt{(1-\xi^2)^2 + \rho^2\xi^2}} \\
&\text{and the phase difference is given by} \\
&\qquad\qquad tg(\gamma - \varphi) = \frac{\rho\,\xi}{1-\xi^2} \\
&\qquad\qquad \text{or} \qquad \sin(\gamma - \varphi) = \rho\,\xi\,F
\end{aligned}\right\} \tag{4}$$

It is very instructive to draw a vector diagram, fig. 2, for the forces expressed in equation (3) by their relative values. They are all considered as outer forces acting on the swing mass like the impulse, and the rotating vectors are shown in their position at t = 0. From this diagram we easily get formulae (4). In the resonance case $\xi = 1$ we have $F = \frac{1}{\rho}$ and $\gamma - \varphi = \frac{\pi}{2}$; The impulse is leading the vibration by $\frac{1}{4}$ period. Disregarding damping outside of resonance we have $F = \left|\frac{1}{1-\xi^2}\right|$ with $\gamma - \varphi = 0$ for $\xi < 1$ and $= \pi$ for $\xi > 1$. The impulse is here acting like an elastic force decreasing or increasing the real elastic force and thereby the frequency.

This is of course well-known, elementary vibration theory, but expressed in the simplest possible way, and a complete familiarity with the terminology is necessary for the following. Furthermore, it is here of some special interest to note, that under steady conditions the vibration phase φ will automatically adjust itself to the value given in (4), which is easily seen from the vector

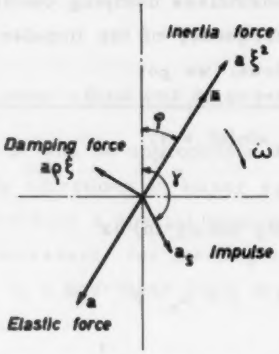


Fig. 2. Vector diagram for steady vibrations.

diagram. If the vibration should lack behind, the resulting elastic force will be bigger, as a bigger component of a_s will act as elastic force. Hence the frequency of the vibration will increase, and the vibration will advance until φ has obtained the correct value. Oppositely, if the vibration by some disturbance momentarily should be too far advanced in relation to the impulse, the frequency will decrease and the vibration fall back to its right position. Thus the phase difference given by (4) is a stable equilibrium.

In a multi-mass, free system fig. 3, consisting of several swing masses on a freely rotating shaft, acted upon by a number of impulses of different size, frequencies and phase angles, the steady vibrations can always be analyzed in their components of the different free vibration forms of the system, of which one is shown in the figure. This is by far not so well-known as the elementary one-mass theory, but nearly indispensable for the correct calculation of any complicated vibration system.

Each of the said vibration components can be calculated exactly after the same formulae as for the one-mass system, when a_s and ρ are specially calculated for each separate vibration form by means of the expressions

$$a_s = \frac{\Sigma(K\,\Delta)}{\omega_e^2\,\Sigma\, m\Delta^2} \quad \text{and} \quad \rho = \frac{\Sigma\, r\,\Delta^2}{\omega_e\,\Sigma m\,\Delta^2} \tag{5}$$

All the masses are namely moving proportionally (in that particular vibration component) with relative amplitudes Δ, and their total inertia is therefore the same as for a single mass $\Sigma m\Delta^2$ placed at the point 1 where Δ is taken as unity. $\Sigma(K\Delta)$ represents the geometric sum of the impulse work vectors (of the same fre-

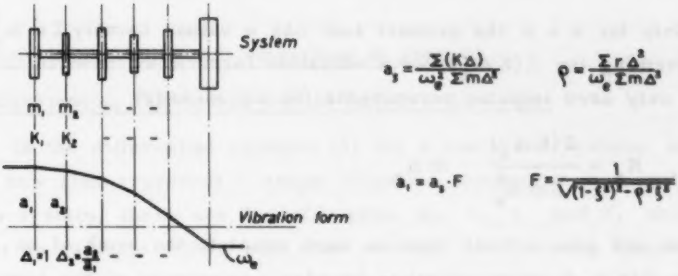


Fig. 3. Linear multi-mass system.

quency) and means a resulting impulse which, working at the point 1, will produce the same impulse work (on that particular vibration component) as all the real impulses of that frequency. Similarly, $\omega_e \Sigma r \Delta^2$ represents the numerical sum of the damping work vectors in the resonance case from which the total relative damping coefficient ρ is derived by the same reduction as used for the impulses. Then the magnification factor and the phase difference between the resulting impulse and the vibration are, for that vibration component, calculated exactly after the formulae (4) set up for a one-mass system.

The given impulses should in the same way be placed on each of the other possible free vibration forms of the system, and the calculated vibrations should all be added (geometrically) giving the real, total vibration, while the calculated vibration components in the separate vibration forms were only abstractions. This may be called the principle of formal superposition, generally not recognized in elementary handbooks although it is easy enough.

The same set of given impulses is thereby used as many times as there are free vibration forms worth while to calculate, but after adding all the calculated vibrations we have in reality only used the full value of the impulses once. This follows from the "orthogonality" of the free vibration forms, i.e. the rule, that for two different vibration forms A and B in a linear system (and also in a non-linear system, "linearized" as explained in section 8) we have always

$$\Sigma m \, \Delta_a \Delta_b = 0$$

Only for a = b the product sum has a value, namely $\Sigma m \Delta_a^2$. When forming the $\Sigma(K\Delta_a)$ for a vibration form A we have in reality only used impulse components (on all masses)

$$K_a = \frac{\Sigma(K\Delta_a)}{\Sigma m \Delta_a^2} \cdot m\Delta_a$$

as these will give a total impulse work equal to the work of an impulse $\Sigma(K\Delta_a)$ while all other impulse components of the form $C \cdot m\Delta_b$ would give an impulse work 0 on the vibration component A. But such a set of "resonance impulses" K_a proportional to $m\Delta_a$ for each mass will always give vibrations of the vibration form A independently of the frequency. On the other hand, adding all the so defined resonance impulses working at an arbitrary point in the system, the resultant must just be the real impulse (possibly no impulse) working at that point, because each resonance impulse is fully used in its own vibration form, so that the total vibration will require the resultant of the resonance impulses, while the given real impulses are, of course, in a unique way producing the total vibration, and so the two sets of impulses must be identical.

When the impulses are at or near resonance with one particular vibration form, it is usually sufficient to consider this vibration component only, calculated by formulae (4) and (5), but it is of importance for the following (and in many other cases) to know, that the method is no accidental approximation but in fact completely correct, if we add the vibration components calculated in the same way for all the other possible free vibration forms of the system (including the free parallel movement, $\Delta = 1$ for all masses, $\omega_e = 0$, $a = -\frac{\Sigma(K)}{\omega^2 \Sigma m}$, disregarding damping).

3. Non-linear crankshaft vibrations in general.

Subharmonic vibrations due to impulse distortion

In the differential equation (1) for a one-mass system, which may now also represent a single vibration component in a multi-mass system, there are 4 coefficients, m, r, s, and K, which might be variable either with time or with the vibration amplitude, each of which possibilities would cause different kinds of non-linearity. But in crankshaft vibrations a variation of r, s, or K does not come into consideration in this connection. r is always so small that a variation of it cannot have any observable influence on the vibrations; s is constant as the vibration stress is always well within the proportionality limit of the shaft material; and K (meaning the maximum value of the impulse) may be variable with the load on the engine but not with time or vibration amplitude at steady load.

Consequently, we should normally have to do only with variation of the swing mass m. Such a variation is always present and of considerable size pr. cylinder due to the variable contribution from the oscillating masses of piston and connecting rod. In practical calculations this variation is usually disregarded and the oscillating masses incalculated as a mean value, but this is in reality a very rough approximation although sufficient for most practical purposes. It is therefore our main purpose here to investigate all consequenses of the mass variation which might lead to important deviations from the common linear theory if only in special cases.

If we now would ask whether this system with variable mass mathematically speaking is a non-linear system or a linear system with variable coefficients (a freely variable system), the answer must be, that freely variable is it only with some approximation, because the inertia moment is depending on the crank angle α, and this is not a pure time function. If the engine is running at a constant number of revolutions n_o, the angle of rotation is $\omega_o t$ at the node point, while the crank angle is $\alpha = \omega_o t + x$. The crankshaft system is therefore, also in mathematical language, a non-linear system although the variation is mainly a time function but it is to some

extent also depending on x, and in our present investigation we cannot allow ourselves to disregard such a dependence without having calculated its importance. We should rather expect this dependence to have some real importance because it represents a physical reality, namely that the variation is "guided" by the vibrating shaft and not freely variable.

Looking now once more at equation (1) to see whether there are other things depending on the parameter α, we find another source of non-linearity, namely the impulse itself. It is calculated by analysis of the tangentialpressure diagram as function of α, and it is solely depending on α as the cylinder pressure is given by the position of the piston. The cylinder impulse K_p of order p periods pr. revolution is therefore in reality:

$$K_p \sin(p\alpha + \gamma) = K_p \sin(p(\omega_o t + x) + \gamma) \cong K_p \sin(p\omega_o t + \gamma) + pxK_p \cos(p\omega_o t + \gamma) \qquad (6)$$

and there is an "artificial impulse" which might be called non-linear, as its size is varying with the vibration amplitude, but if we place it at the left side of the differential equation, it is only a "linear" elasticity with variable coefficient (Matthieu's equation) - which shows better than anything else, that a variable linearity might as well be called a non-linearity.

We have, accordingly, in the crankshaft system two sources of non-linearity, namely impulse distortion and mass variation. The latter will be our main subject, but we shall first look a little at the former.

Although the artificial impulse (6) is always very small, as $px \ll 1$, it has a general effect of some importance, namely its ability in the resonance case to pick up energy from the pure rotation of the shaft and transform it into vibration energy. Considering first a one-cylinder engine it is seen, that when the impulse $K_p \sin(p\alpha + \gamma)$ is in resonance with the frequency of the crankshaft, the very vibration which it excites itself, $x = -a\cos(p\omega_o t + \gamma)$, will produce an artificial impulse:

$$- paK_p \cos^2(p\omega_o t + \gamma) = -\frac{1}{2} paK_p - \frac{1}{2} paK_p \cos(2p\omega_o t + 2\gamma) \quad (7)$$

The new impulse of order 2p is usually without importance, only producing small forced vibrations, but the constant "impulse", $-\frac{1}{2}paK_p$, means a braking momentum which, when surmounted by the power of the engine, will pick up a work

$$\pi paK_p \quad \text{pr. rev.} \quad (8)$$

equal to the work done by the main impulse $K_p \sin(p\omega_o t + \gamma)$ on the vibration $- a \cos(p\omega_o t + \gamma)$ during p vibrations, and of course also equal to the damping work which is

$$\pi pra\omega_e a = \pi p\rho a^2 \cdot m\omega_e^2 = \pi paa_s \cdot m\omega_e^2 = \pi paK_p$$

The cylinder impulses can excite the vibrations but cannot themselves supply vibration energy, as they are merely acting like elastic springs. But through the impulse distortion a braking of the shaft and an energy supply to the vibrations are effected at the same time, because the impulses considered as springs are not worked from outside, but from the rotating shaft itself. The impulses are "guided" by the vibrating shaft.

Considering next a multi-cylinder engine at resonance in vibration form A with the cylinder impulses of order p, the vibration synchronous for all cylinders may be $a \cdot \sin(p\omega_o t + \varphi)$ at cyl. 1 and $x = a\,\Delta \sin(p\omega_o t + \varphi)$ at the other cylinders, while the impulse is $K_p \sin(p\alpha + \gamma)$ at cyl. 1 and $K_p \sin(p(\alpha + \theta) + \gamma)$ at another cylinder with crank angle θ in relation to cyl. 1. It is then seen, that while the impulse work vectors for the main impulses are of a size proportional to $K_p \Delta$ and situated at relative angles $p\theta$, the artificial impulses are already in themselves of a size proportional to $K_p \Delta$ and situated at the same relative angles. But the constant part of the artificial impulses (7) are braking momentums and should be added directly, consequently by means of the same vector diagram as used to calculate $a_s = \frac{\Sigma (K\Delta)}{\omega_e^2 \Sigma m\Delta^2}$. Therefore, as the total impulse

acting on the system considered as one-mass system is $a_s \cdot \omega_e^2 \Sigma m \Delta^2$, the total braking will be $-\frac{1}{2} p a a_s \cdot \omega_e^2 \Sigma m \Delta^2$ giving the correct energy supply.

It is seen that any impulse of order p will (in multi-cylinder as well as one-cylinder engines) under the influence of a big resonance vibration of order q give artificial impulses of order p-q and p+q, but normally they will not themselves be in resonance with the vibration and will therefore be very harmless. The only dangerous case is

$$p - q = q \qquad\qquad p = 2q$$

in which case we may get a subharmonic vibration, a resonance vibration excited by an impulse of double the frequency of the vibrations.

We shall first consider this case in a one-cylinder engine. It is convenient here to use the differential equation in the form (2). In stead of a_s we shall, however, write a_u meaning exactly the same for a one-cylinder engine, namely $\frac{K}{m\omega_e^2}$, but for a multi-cylinder engine it will be found different from a_s, as we shall see. For a one-cylinder engine with a relative impulse a_u or a_s of order p = 2q and in resonance with q vibrations pr. rev. we have then:

$$\begin{aligned}\frac{d^2x}{\omega_e^2 dt^2} + \rho \frac{dx}{\omega_e dt} + x &= a_u \sin(p\alpha + \gamma) = a_u \sin(2q(\omega_o t + x) + \gamma) \\ &= a_u \sin(2\omega_e t + \gamma) \cdot \cos 2qx + a_u \cos(2\omega_e t + \gamma) \cdot \sin 2qx \\ &\cong a_u \sin(2\omega_e t + \gamma) \cdot (1 - 2q^2x^2) + a_u 2qx \cdot \cos(2\omega_e t + \gamma) \\ &= a_u \sin(2\omega_e t + \gamma) + 2qa_u x \cdot \cos(2\omega_e t + \gamma) - 2q^2 a_u x^2 \sin(2\omega_e t + \gamma)\end{aligned} \tag{9}$$

this time using 3 members in the series for the sinus of a big plus a small angle, so that we have both a primary and a secondary artificial impulse.

Introducing $x = a \sin(\omega_e t + \varphi)$ in the primary artificial impulse (2. member in the last line of eq. (9)) we get from it a part of order q, that means an impulse of angular velocity $q\omega_o = \omega_e$, in resonance with the vibration. This impulse will be:

$$a_q' = - q \, a_u a \cdot \sin(\omega_e t + \gamma - \varphi) \tag{10}$$

It is essential that the phase of the artificial impulse is depending on the vibration phase φ but with negative sign. If it had been positive sign, the phase difference between impulse and vibration would have been unchanging, but as it is, the vibration is free to adjust itself to any suitable phase difference $\gamma - \varphi - \varphi = \gamma - 2\varphi$ between impulse and vibration, just as in ordinary resonance case explained in connection with fig. 2. In the present case eq. (9) concerns strict resonance, $\xi = 1$, with ξ meaning $\frac{q\omega_o}{\omega_e}$, and the said phase difference will tend (as the impulse (10) is negative) towards

$$\gamma - 2\varphi = -\frac{\pi}{2} \qquad \varphi = \frac{\pi}{4} + \frac{\gamma}{2}$$

in which case the impulse is directly opposed to the damping force, giving the condition for a steady vibration:

$$\rho a = q a_u a \qquad \rho = q a_u \tag{11}$$

While in usual resonance the condition $\rho a = a_s$ gives a determination of the amplitude a, the corresponding condition (11) in the subharmonic case leaves the amplitude underdetermined and yields in stead of it a value for the damping which will allow steady vibrations of arbitrary size. If the damping is bigger than given by (11), the vibration cannot come up, and if the damping is smaller, the vibration will immediately arise and increase to the utmost limits or breaking-down of the system.

If there were no damping, $\rho = 0$, the condition for a steady solution would be that the impulse (10) is either in phase with the vibration or directly opposed to it, $\gamma - 2\varphi = 0$ or π, in which case (10) would be an elastic force, either increasing or decreasing the frequency, so that we have in a way a "splitting-up" of the proper frequency in 2 resonance points (numbers of revs.) where steady vibrations can exist of arbitrary size, but between them there will be a range with increasing vibrations (in the mathematical language called "instability" - but nobody would expect "stability" at resonance when the impulse is bigger than the damping).

Writing (9) in the form (3) we find the limits of the dangerous range given, analogous with (11), by

$$(1 - \xi^2) \cdot a = \pm q a_u a \qquad \xi \cong 1 \pm \frac{1}{2} q a_u$$

In the general case with a damping $\rho < q a_u$ we get

$$(1 - \xi^2)^2 + \rho^2 \xi^2 = (q a_u)^2 \qquad \xi \cong 1 \pm \frac{1}{2} \sqrt{(q a_u)^2 - \rho^2} \tag{12}$$

a result which is also found by putting the ordinary magnification factor (4) equal to $\frac{a}{qa_u a} = \frac{1}{qa_u}$. F is namely the magnification factor for steady vibrations and in the present case valid only, when there <u>are</u> steady vibrations.

We have calculated the secondary artificial impulse in (9) in order to show, that as the vibration is here excited by the primary artificial impulse, it is the secondary that will produce the braking momentum supplying the energy. Introducing $x = a \sin(\omega_e t + \frac{\pi}{4} + \frac{\gamma}{2})$ in the last member in (9) we find a constant part

$$- 2q^2 \cdot a_u a^2 \cdot \frac{1}{2} \cdot \frac{1}{2} = - \frac{1}{2} q^2 a^2 a_u$$

and a braking work pr. rev.

$$\pi q^2 a^2 a_u \cdot m\omega_e^2 = \pi q^2 a^2 K_p$$

equal to the work pr. q vibrations of the impulse qaK_p and also equal to the damping work in case of the equilibrium condition (11).

Comparing with (8) it is seen that the impulse work in subharmonic excitation is much smaller due to the extra factor $a << 1$, but at the same time of a much more dangerous nature as it is increasing with 2. power of the amplitude like the damping work, and the vibrations are therefore not limited by damping, if they can arise at all.

The differential equation (9) and consequently also the main results (11) and (12) will be correct also for multi-cylinder engines when the relative value a_u of the forcing impulse (of order $p = 2q$) is calculated as

$$a_u = \frac{\Sigma(K_p \Delta^2)}{\omega_e^2 \Sigma m \Delta^2} \qquad (13)$$

in which $\Sigma(K_p\Delta^2)$ means the usual vector summation for vectors of order p and relative angles $p\theta$ but of size $K_p\Delta^2$ in stead of $K_p\Delta$ as used for a_s. The primary artificial impulses from the different cylinders are namely of the form $x \cdot K_p \sin p(\alpha + \theta)$, and as x is synchronous for all cylinders, it cannot change the relative phase angles, only reduce the size with the factor Δ. Consequently the impulse vectors are proportional to $K_p\Delta$ and the impulse work vectors proportional to $K_p\Delta^2$, but at the same angles as the ordinary impulses. For a point-symmetric vibration form $\Sigma(K\Delta^2)$ is very different from $\Sigma(K\Delta)$, and the order numbers giving the biggest impulse summation for subharmonic excitation may therefore be different from those giving the biggest ordinary excitation.

In practice the possibility of such vibrations is small (otherwise we should have heard more about them!), because qa_u will usually be smaller than the coefficient of natural damping inherent in the crank mechanism itself, which is usually of an order of size ρ = ab. 0,02, and besides there may be additional damping from the device driven by the engine. a_u will be sufficiently big only if it is due to the biggest possible impulse and even then probably only in 2 special cases, namely a 4-cyl. engine with all cranks in a plane (K_2 giving subharmonic response at $q = 1$, whether the vibration form is symmetric or not) and a 6-cyl. engine with symmetric crank position in triangle (K_3 at $q = 3/2$). The direct resonance with this big impulse (although depending on another impulse summation, as mentioned) should then be situated below half of the maximum revolutions, as the subharmonic will occur at the double speed, and the system must therefore have such an unusually low frequency as can probably only happen (in stationary engines) when there is an elastic coupling between engine and driven device. In such a case, however, the danger of getting these vibrations is completely avoided by providing the coupling itself with a certain (not very big) inner damping - but of course, it can be important to know this, because the undamped subharmonic vibrations are of such a violent nature as to make it unbelievable, that they could be completely eliminated by a little damping [1].

In ship engines, where the one-node vibration form may have a low frequency due to a long intermediate shaft, there will in this vibration form be a very big damping from the propeller, so that there is normally no danger of subharmonic vibrations in these cases.

A common school example of subharmonic vibrations due to impulse distortion is the mathematical pendulum (mass M, length L, $\omega_e^2 = g/L$ for small vibrations) for which the suspension point is given a vertical movement $y = f \sin 2\omega t$. The vertical impulse acting on the pendulum mass is then:

$$P = -M\frac{d^2y}{dt^2} = 4fM\omega^2 \sin 2\omega t$$

having a horizontal component $P \sin x \cong Px$ for small angular amplitudes x. Disregarding damping (which ought to be negligible in a pendulum) the differential equation for small vibrations is

$$ML\frac{d^2x}{dt^2} + Mgx = 4fM\omega^2 \sin 2\omega t \cdot x$$

---

[1]) In his contribution to litt. 6 the author has given a description of such a case.

and in reduced form

$$\xi^2 \frac{d^2x}{d(\omega t)^2} + x = \frac{4f}{L}\,\xi^2 \cdot \sin 2\omega t \cdot x$$

$$\text{with } \xi = \frac{\omega}{\omega_e} \qquad \text{and } \omega_e^2 = \frac{g}{L}$$

Introducing $x = a \sin(\omega t + \varphi)$ we get a 1. order artificial impulse of size $a \cdot \frac{2f}{L} \cdot \xi^2$, free to adjust itself in phase. It is a case of subharmonic excitation as the frequency of the vibration is half of that of the forcing (here vertical) impulse. For $\xi = 1$ the artificial impulse will be free to produce increasing vibrations. For ξ somewhat more or less than 1 the artificial impulse will act as elastic force, giving steady vibrations for

$$-\xi^2 + 1 = \pm \frac{2f}{L} \cdot \xi^2 \qquad \xi^2 = \frac{1}{1 \pm \frac{2f}{L}} \sim 1 \pm \frac{2f}{L}$$

There will thus be instability (increasing vibrations) in a range approximately

$$1 - \frac{f}{L} < \xi < 1 + \frac{f}{L} \qquad \text{(assuming } f << L\text{)}$$

and outside this range free vibrations will be decreasing, as the artificial impulse will here act like a damping.

The amusing point is, that with the pendulum turned upside down the vertical top position may be a stable equilibrium. Measuring x out from this position the differential equation is now

$$\xi^2 \frac{d^2x}{d(\omega t)^2} - x = -\frac{4f}{L} \cdot \xi^2 \cdot \sin 2\omega t \cdot x$$

To get steady vibrations the artificial impulse must here necessarily be negative, and we have only one possibility:

$$-\xi^2 - 1 = -\frac{2f}{L} \cdot \xi^2 \qquad \xi^2 = \frac{L}{2f - L} \qquad \text{requiring } f > \frac{1}{2} L$$

For ξ smaller than given by this formula the artificial impulse is too small to counteract both elastic force and inertia force; there will be increasing vibrations and the pendulum will quickly fall down. But for ξ bigger than this value, there is stability as the free vibrations will be decreasing. Solving the equilibrium condition for the vertical movement f we get

$$f = L \frac{1 + \xi^2}{2\xi^2}$$

and it is seen that the vertical movement required for stability is rather big. For $\xi = 1$ (which means that the frequency of the vertical movement is double of the natural frequency of the pendulum in ordinary, down-hanging position) we must have $f > L$, and even for the highest possible frequency of the vertical movement we shall have $f > \frac{1}{2} L$.

4. Variable mass in one-cylinder and multi-cylinder engines

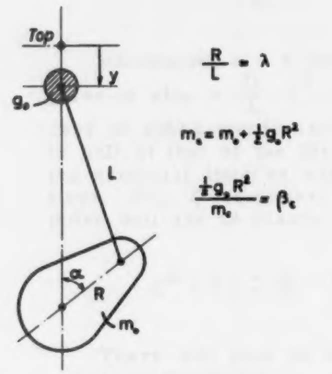


Fig. 4. Crank mechanism.

In the crank mechanism fig. 4 the piston movement is with good approximation

$$y = R(1-\cos\alpha) + \frac{1}{4}\lambda R(1-\cos 2\alpha)$$

and $\frac{dy}{d\alpha} = R(\sin\alpha + \frac{1}{2}\lambda \sin 2\alpha)$

The inertia moment of the purely rotating parts is called m_o, and for a crank angle α the contribution of the oscillating mass g_o to the inertia moment of the crank is:

$$g_o R^2 \left(\frac{dy}{R d\alpha}\right)^2 = g_o R^2 \cdot (\sin\alpha + \frac{1}{2}\lambda \sin 2\alpha)^2 = g_o R^2 \cdot (\sin^2\alpha + \lambda \sin\alpha \sin 2\alpha + \frac{1}{4}\lambda^2 \sin^2 2\alpha)$$

$$= \frac{1}{2} g_o R^2 \cdot (1 + \frac{1}{4}\lambda^2) + \frac{1}{2} g_o R^2 \cdot (\lambda\cos\alpha - \cos 2\alpha - \lambda\cos 3\alpha - \frac{1}{4}\lambda^2 \cos 4\alpha)$$

We shall disregard the small term $\frac{1}{4}\lambda^2$. The constant member $\frac{1}{2} g_o R^2$ is included in the mean inertia moment $m_c = m_o + \frac{1}{2} g_o R^2$, and the proportion $\frac{\frac{1}{2} g_o R^2}{m_c}$ is called β_c. The total inertia moment pr. cylinder is then

$$J_c = m_c (1 - \Sigma \beta_{pc} \cos p\alpha) \tag{14}$$

with $\beta_{pc} = \beta_c(-\lambda, 1, \lambda)$ for $p = (1, 2, 3)$

There is a principal variation twice pr. revolution and two smaller variations one and three times pr. rev. Variation components of order higher than 3 may safely be disregarded in practical calculations.

Besides the mass variation the oscillating mass is producing an outer impulse acting on the crankshaft, and when fully considering these two effects active in the vibration system, we do not need to think any more of the oscillating masses themselves. At steady speed of revolution ω_o this outer impulse pr. cylinder can be calculated (as will be seen in the following) as:

$$-\frac{1}{2}\frac{dJ_c}{d\alpha}\left(\frac{d\alpha}{dt}\right)^2 = -\frac{1}{2}m_c\omega_o^2\,\Sigma p\beta_{pc}\sin p\alpha = \Sigma K_{po}\sin p\alpha \qquad (15)$$

in which latter expression the coefficients

$$K_{po} = g_o R^2\omega_o^2\left(\frac{1}{4}\lambda,\ -\frac{1}{2},\ -\frac{3}{4}\lambda\right) \quad \text{for } p = (1, 2, 3)$$

are the common expressions for the impulses from the oscillating masses, usually added directly to the cylinder impulse K_p of same order, as these big impulses are sinus components with a very small phase angle γ at the top position. Strictly speaking, we should add K_p and K_{po} as vectors with phase difference γ.

Dividing (15) with $m_c\omega_e^2$ and using $q = \frac{\omega_e}{\omega_o}$ we get the impulses from the oscillating mass expressed in terms of static amplitudes as

$$a_{po} = -\frac{p}{2q^2}\beta_{pc} \qquad (16)$$

In multicylinder engines the analysis in free vibration forms, explained in section 2, will be correct also when taking the true, variable contributions from the oscillating masses into consideration. These masses are stiffly connected with the constant swing masses and do not increase the number of separate masses characterizing the system. There is still the same number of free vibration forms, and they have the same shape as for constant masses, as they should (by definition) be calculated for the mean values of the variable masses. They have also definite proper frequencies, not changing with the size of the vibrations, although (as we shall see) slightly higher than for constant masses.

While in linear multi-mass systems the different free vibration components are completely independent of each other, so that each of them can exist separately, there may in the variable-mass system be a certain, usually very weak coupling between different vibration components (this matter is treated in section 10). But in

each single vibration form A the action of the variable masses back upon the vibration component A itself must be the same as the action from the variation of the total $\Sigma m \Delta^2$, this variation being imagined equally distributed over all the masses. If it is not so in reality, the inequality will have some effect on other vibration components, thereby causing the said coupling effect, but not changing vibration form A itself.

If α is the crank angle of cyl. 1, measured from top, its variable contribution of order p (p = 1, 2 or 3) to the total $\Sigma m \Delta^2$ is $-\beta_{pc} m_c \cos p\alpha$, and the contribution of another crank at angle θ in relation to crank 1 is $-\beta_{pc} m_c \Delta^2 \cos p(\alpha + \theta)$. It follows that the variable contributions from all cylinders are given as p-order vectors of size $\beta_{pc} m_c \Delta^2$ and relative angles $p\theta$, giving a resulting variation coefficient

$$\beta_p = \frac{\Sigma(\beta_{pc} m_c \Delta^2)}{\Sigma m \Delta^2} \tag{17}$$

In the equivalent one-mass system representing a single vibration form in a multi-cylinder engine the total, variable inertia moment will then be

$$J = m\,(1 - \Sigma\beta_p \cos p(\alpha + \mu)) \tag{14a}$$

with m standing for $\Sigma m \Delta^2$ and $p\mu$ meaning the phase angle of vector β_p when α is still measured from crank 1 in top.

In the following we shall mainly treat the 2. order component of the mass variation and for convenience write β without index in stead of β_2. Furthermore, we shall omit the deviation angle μ, now measuring α out from the "main position" of the shaft, i.e. the position at which vector β is at top and the total inertia moment J is minimum. We have then

$$J = m\,(1 - \beta \cos 2\alpha) \tag{14b}$$

We can also consider the equivalent system as a one-cylinder engine with $\lambda = 0$ and an oscillating mass

$$g_o' = g_o \frac{\beta m}{\beta_o m_o} \quad , \quad \frac{1}{2} g_o' R^2 = \beta m \tag{14c}$$

Fig. 5 shows the vector summation for the 2. order mass variation in vibration form A for the 10-cylinder engine mentioned in the introduction, and the whole calculation necessary is shown in the following table, in which all inertia moments are given as relative values in proportion to the inertia moment of cyl. 1. β_c was 0,31 for all cylinders, and the resulting β = 0,11. The

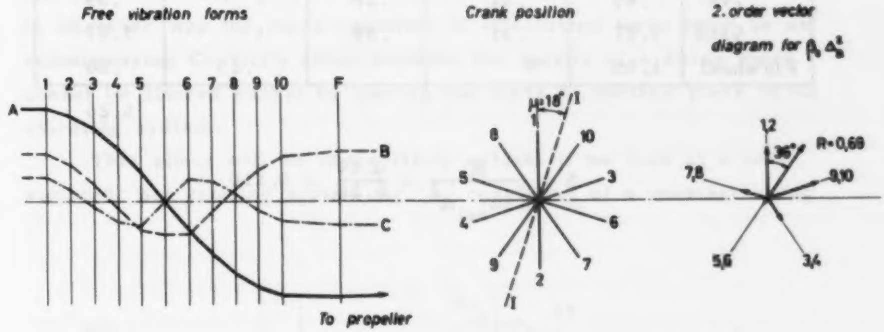


Fig. 5. Variable mass in 10-cylinder engine.

main position, marked I, is at μ = 18° from crank 1, which also follows from the symmetry of the crank position.

β for vibration form A

	Δ^2 (2 cyl.)	β_c (pr. cyl.)	$\beta_c \Delta^2$	m_{rel} (pr. cyl.)	$m_{rel} \cdot \Delta^2$
Cyl. 1+2	1.85	.31	.57	1	1.85
3+4	.93	.31	.29	1	.93
5+6	.19	.31	.06	2.40	.46
7+8	.89	.31	.28	1	.89
9+10	1.87	.31	.58	1	1.87
Flywheel	1.05	0		.25	.26
					6.26

$$\beta = \frac{R}{\Sigma\, m_{rel} \Delta^2} = \frac{0.69}{6.26} = 0.110$$

5. System with evident Corioli's force

In the first instance one might suppose that the differential equation for the now considered variable one-mass system could be derived from equation (1) simply by substituting m with $m(1 - \beta \cos 2\alpha)$, but this would not be correct because there must necessarily appear a Corioli's force acting on the so defined system as an outer force, and this must require a new member in the differential equation. In whatever way the mass variation is effectuated there must be an accompanying Corioli's effect because the inertia of a swing mass cannot be altered except by moving the mass to another place in the vibrating system.

This effect will be immediately evident if we look at a constructed, not rotating system fig. 6, consisting of a constant swing

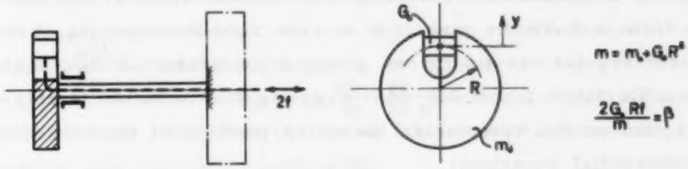


Fig. 6. Constructed system to show Corioli's force.

mass m_o and an additional mass G_o moving in a radial slot and driven (as suggested) by an outer mechanism, independent of the vibrations. In order to make comparison with the crank mechanism the movement of G_o, measured positively outwards from the mean radius R, shall be defined as

$$y = - f \cos 2\omega_o t$$

and the total, momentary inertia moment will be

$$J = m_o + G_o(R + y)^2$$

The differential equation for the vibrations in such a system is generally (see f. inst. Timoshenko, litt. 8) found by means of Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q$$

Using x as parameter we have $T = \frac{1}{2} J \dot{x}^2$ where J is here time-depending only (through y), consequently:

$$\frac{\partial T}{\partial \dot{x}} = J\dot{x} \qquad \frac{\partial T}{\partial x} = 0$$

and Lagrange's equation means that $m\frac{d^2x}{dt^2}$ in equation (1) should be substituted with

$$\frac{d(J\dot{x})}{dt} = J\frac{d^2x}{dt^2} + \frac{dJ}{dt} \cdot \frac{dx}{dt} \tag{18}$$

in which the second member clearly has the character of the momentum from a Corioli's force. It is seen directly from fig. 6 that besides the regular inertia forces giving a momentum $J\frac{d^2x}{dt^2}$ there must be a Corioli's force $2G_o \frac{dy}{dt} \cdot \frac{dx}{dt}$ giving a momentum (acting as a reaction on the system and therefore positive at the left side of the differential equation):

$$2G_o(R + y)\frac{dy}{dt} \cdot \frac{dx}{dt} = \frac{dJ}{dt} \cdot \frac{dx}{dt}$$

When f << R we have

$$J \cong m_o + G_oR^2 - 2G_oRf\cos 2\omega_o t = m(1 - \beta\cos 2\omega_o t)$$

with $m = m_o + G_oR^2$ and $\beta m = 2G_oRf$

Expression (18) is then

$$m(1 - \beta\cos 2\omega_o t) \cdot \frac{d^2x}{dt^2} + 2\beta m\omega_o \sin 2\omega_o t \cdot \frac{dx}{dt} \tag{19}$$

and when there is also an outer impulse $K \sin(\omega t + \gamma)$ acting directly on the swing mass, the complete differential equation will be:

$$m(1-\beta \cos 2\omega_o t)\cdot \frac{d^2x}{dt^2} + 2\beta m\omega_o \sin 2\omega_o t \cdot \frac{dx}{dt} + r\frac{dx}{dt} + sx = K \sin(\omega t + \gamma) \qquad (20)$$

As for the driving mechanism for the moving mass, this will be acted upon not only by the regular inertia force $-G_o\frac{d^2y}{dt^2}$ (which is without interest and could be equalized by a spring), but also by a special centrifugal force

$$C = G_o R\left(\frac{dx}{dt}\right)^2 \qquad (21)$$

which may consume work and certainly will do so in the resonance case.

Perhaps the comparison with the crank mechanism would be more obvious, if we let G_o have the same size as g_o in the crank mechanism and let it slide all along a centerline, its distance from the center being $y = R \sin\omega_o t$, thereby giving a direct impression of the oscillating mass g_o's participation in the inertia moment. The differential equation would be exactly the same as (20). But even such a system would not, as we shall see, in all respects be equivalent with the crank mechanism, as it is a "freely variable" system, the mass variation being a pure time-function and not influenced by the vibrations. Equation (20) is in mathematical terminology a "linear" equation with variable coefficients.

6. Correct differential equation for the crank mechanism

The common crank mechanism fig. 4 is functioning at the same time as power drive, pulsator for vibrations and driving mechanism for a variable mass, while the two latter functions are separated in the system fig. 6. Accordingly, we shall in the correct differential equation for the crank mechanism find members not alone corresponding to the Corioli's force in (19) but also to the reaction from the piston rod in its capacity as driving mechanism for a variable mass.

Using the crank angle α as parameter in Lagrange's equation (see also Malkin, litt. 9) we have $T = \frac{1}{2} J \dot{\alpha}^2$ with

$$J = m(1 - \beta \cos 2\alpha) \qquad \text{and} \qquad \alpha = \omega_o t + x$$

consequently

$$\frac{\partial T}{\partial \dot{\alpha}} = J\dot{\alpha} \qquad\qquad \frac{\partial T}{\partial \alpha} = \frac{1}{2}\frac{dJ}{d\alpha} \cdot \dot{\alpha}^2$$

and Lagrange's equation means now, that $m\frac{d^2x}{dt^2}$ in equation (1) should be substituted with

$$\frac{d(J\dot{\alpha})}{dt} - \frac{1}{2}\frac{dJ}{d\alpha} \cdot \dot{\alpha}^2 = J\frac{d^2\alpha}{dt^2} + \frac{dJ}{dt} \cdot \frac{d\alpha}{dt} - \frac{1}{2}\frac{dJ}{d\alpha} \cdot \left(\frac{d\alpha}{dt}\right)^2 = J\frac{d^2x}{dt^2} + \frac{1}{2}\frac{dJ}{d\alpha}\left(\frac{d\alpha}{dt}\right)^2$$

$$= m(1 - \beta \cos 2\alpha) \cdot \frac{d^2x}{dt^2} + \beta\, m \sin 2\alpha \cdot \left(\omega_o + \frac{dx}{dt}\right)^2 \tag{22}$$

in which the 2. member contains the same Corioli's momentum as in (19) (when approximating α with $\omega_o t$) although it is difficult to perceive it as such in the crank mechanism. Besides, this 2. member contains a constant times $\sin 2\alpha$, which is the regular 2. order impulse from the oscillating mass (the same as (15) for $p = 2$), and a term with $\left(\frac{dx}{dt}\right)^2$, reminding of the centrifugal force (21).

The expression (22), meaning the total reaction from the moving masses, could also be found directly, without using Lagrange's theory, namely by going back to the fundamental equation for the movement of the oscillating mass, when $\lambda = 0$,

$$y = R\,(1 - \cos\alpha)$$

and already here introduce $\alpha = \omega_o t + x$. We then get:

$$\frac{dy}{dt} = R\sin\alpha \cdot (\omega_o + \frac{dx}{dt}) \qquad \frac{d^2y}{dt^2} = R\cos\alpha \cdot (\omega_o + \frac{dx}{dt})^2 + R\sin\alpha \cdot \frac{d^2x}{dt^2}$$

The oscillating mass is g'_o by (14 c) in the general case of an equivalent one-cylinder engine, and the reaction from the inertia force $- g'_o \cdot \frac{d^2y}{dt^2}$ on the vibration system is then:

$$g'_o \frac{d^2y}{dt^2} \cdot R\sin\alpha = \frac{1}{2} g'_o R^2 (\sin 2\alpha \cdot (\omega_o + \frac{dx}{dt})^2 + (1 - \cos 2\alpha) \cdot \frac{d^2x}{dt^2})$$

which together with the reaction from the constant swing mass, $m_o \frac{d^2x}{dt^2}$, and with $\frac{1}{2} g'_o R^2 = \beta m$, gives a total momentum on the system:

$$m\,(1 - \beta\cos 2\alpha) \cdot \frac{d^2x}{dt^2} + \beta m \sin 2\alpha \cdot (\omega_o + \frac{dx}{dt})^2 \;,$$

the same as (22). Having now got a double security in the argumentation we can with confidence write the complete differential equation as:

$$m\,(1 - \beta\cos 2\alpha) \frac{d^2x}{dt^2} + \beta m \sin 2\alpha \cdot (\omega_o + \frac{dx}{dt})^2 + r\frac{dx}{dt} + sx = K\sin(p\alpha + \gamma) \qquad (23)$$

Remembering that α is not a pure time variable, but $= \omega_o t + x$, the equation is indeed very complicated, but we shall see in the following, that simple and yet in 1. degree completely correct approximations can be made for all practical demands.

For a full understanding of the action of the various forces it will be of some interest also to compare with a rotating system as fig. 6, the fixation being substituted with a heavy flywheel rotating with constant angular velocity ω_o. We shall now use $\alpha = \omega_o t + x$ as parameter, but as J is still time-depending only, there will be no $\partial T / \partial\alpha$, and we have in expression (18) only to substitute dx/dt with $d\alpha/dt$, whereby the differential equation (20) changes into:

$$m(1 - \beta\cos 2\omega_o t) \cdot \frac{d^2x}{dt^2} + 2\beta m\omega_o \sin 2\omega_o t \cdot (\omega_o + \frac{dx}{dt}) + r\frac{dx}{dt} + sx = K\sin(\omega t + \gamma) \qquad (24)$$

thus merely adding a 2. order impulse from the variable and now rotating mass. This impulse will be (when transferred to the right side of the equation)

$$- 2\beta m \omega_o^2 \sin 2\omega_o t$$

while it is only half as big in the crank mechanism with the same mass-variation factor β, namely by (23):

$$- \beta m \omega_o^2 \sin 2\alpha$$

The reason for the difference is, that in the crank mechanism the regular driving of the oscillating mass reduces the "Corioli's effect" (if we may call it so in the crank movement), while in the freely variable and rotating system fig. 6 the driving mechanism for G_o is independent of the rotation. In order to obtain complete similarity between a rotating system fig. 6 and the crank mechanism, the moving mass G_o should be guided by a fixed leading curve as suggested in fig. 7, so that the position of G_o is determined directly by the "crank angle" $\alpha = \omega_o t + x$, the movement of G_o being $y = - f\cos 2\alpha$. The differential equation for such a system is completely the same as (23), perhaps with exception of the outer impulse K which might still be a pure time-function, although it is difficult to see how it could be transmitted to a rotating system except through the tu rning angle α.

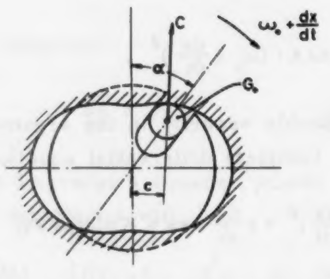


Fig. 7. Rotating variable-mass system with fixed guiding.

It is now seen, that the Corioli's momentum:

$$2G_oR\frac{dy}{dt}\cdot\frac{d\alpha}{dt} = 2G_oR\frac{dy}{d\alpha}\left(\frac{d\alpha}{dt}\right)^2 = 2\beta m \sin 2\alpha \cdot \left(\frac{d\alpha}{dt}\right)^2$$

is partly counteracted by the reaction from the roller pressure, which is, for $f << R$, equal to the total centrifugal force acting on G_o, $C = G_oR\left(\frac{d\alpha}{dt}\right)^2$, and has a centerdistance $c = \frac{dy}{d\alpha} = 2f\sin 2\alpha$, so that its reaction (measured positively against the rotation) is

$$- G_oR\left(\frac{d\alpha}{dt}\right)^2 \cdot 2f\sin 2\alpha = - \beta m \sin 2\alpha \left(\frac{d\alpha}{dt}\right)^2$$

which is just the term $-\frac{\partial T}{\partial \alpha}$ in Lagrange's equation, not present when J is independent of the chosen parameter. When J is depending on a parameter α, then it must necessarily mean that there is a stiff connection between the variable mass and the part directly represented by the parameter, consequently also a reaction between them, and from this we get the term $-\frac{\partial T}{\partial \alpha} = -\frac{1}{2}\frac{dJ}{d\alpha}\cdot\dot{\alpha}^2$, opposed to and half as big as the term $\frac{dJ}{dt}\cdot\dot{\alpha}$ coming from the first member in Lagrange's equation, $\frac{d}{dt}(J\dot{\alpha})$, and representing the Corioli's effect. Thus the different vibration systems here described give a nice illustration of Lagrange's ingenious theory.

In the freely variable and rotating system fig. 6 the Corioli's momentum is, by (24):

$$2\beta m\omega_o(\omega_o + \frac{dx}{dt})\cdot \sin 2\omega_o t$$

while the combined Corioli's momentum and reaction from the driving mechanism for the variable mass is, by (23), both for the guided system fig. 7 and for the crank mechanism (where they cannot be separated):

$$\beta m(\omega_o + \frac{dx}{dt})^2 \cdot \sin 2\alpha$$

The regular 2. order impulse is in the latter case reduced to the half, $-\beta m\omega_o^2 \sin 2\alpha$, but the Corioli's momentum from the vibrations is practically the same in both cases, $2\beta m\omega_o \frac{dx}{dt}$ times $\sin 2\omega_o t$ or $\sin 2\alpha$, and the vibrations are therefore also practically the same, when not specially exited by the 2. order impulse. Finally we have in the latter case the member containing $(dx/dt)^2$ analogous with the centrifugal force (21) acting independently in the freely variable system.

7. Free vibrations

To solve equation (23) for free vibrations we shall disregard damping and outer impulse, so that the equation is

$$m(1-\beta\cos 2\alpha)\cdot\frac{d^2x}{dt^2}+\beta m\sin 2\alpha\cdot\left(\omega_o+\frac{dx}{dt}\right)^2+sx=0 \qquad (25)$$

The first question is, whether it will be allowable in the equation to substitute α with the pure time variable $\omega_o t$. Seeking a 1. degree approximation which will be completely correct for small values of the vibration amplitude a, we need only to take regard to all members in (25) which will contain a in 1. power, namely containing x, $\frac{dx}{dt}$ or $\frac{d^2x}{dt^2}$ as single factor. Now $\cos 2\alpha$ is in reality, analogous with the impulse distortion mentioned in section 3:

$$\cos 2\alpha = \cos 2\omega_o t - 2x\sin 2\omega_o t$$

and as the correction member contains x and should be multiplied with $\frac{d^2x}{dt^2}$, we can here disregard it. In the second member in (25) we can first disregard $\left(\frac{dx}{dt}\right)^2$, next we can disregard the angle correction of $\sin 2\alpha$ in connection with the factor $2\omega_o\frac{dx}{dt}$, but strictly speaking we should not disregard the angle correction in connection with the factor ω_o^2, that means in reality the ordinary impulse distortion of the 2. order impulse from the oscillating masses, which distortion would be (as an impulse)

$$-2\beta m\omega_o^2 x\cdot\cos 2\omega_o t \qquad (26)$$

accordingly of 1. power in a. But, as we shall see, this contribution will be comparatively small for usual values of q (the number of vibrations pr. revolution), and we shall so far disregard it (but not forget it completely). Finally we can here disregard the main 2. order impulse in the same way as we have disregarded all the cylinder im-

pulses which will always be present but generally give forced vibrations only, which have nothing to do with free vibrations.

The differential equation is now

$$m(1-\beta\cos 2\alpha)\cdot\frac{d^2x}{dt^2}+2\beta m\omega_o\sin 2\alpha\cdot\frac{dx}{dt}+sx=0 \qquad (27)$$

in which α may be reckoned as $\omega_o t$. This being so, we shall from now on use $\alpha = \omega_o t$ as independent variable. Strictly speaking we ought perhaps to introduce a new letter for it, but there will be no mistakes in using the familar α now as a regular time variable, when we remember, that whenever greater accuracy is wanted, we must go back to the complete differential equation (23) or (25) and here introduce the α-corrections that might be necessary.

Dividing (27) with $m\omega_o^2$ and using $q = \frac{\omega_e}{\omega_o}$ we finally get a reduced form

$$(1-\beta\cos 2\alpha)\cdot\frac{d^2x}{d\alpha^2}+2\beta\sin 2\alpha\cdot\frac{dx}{d\alpha}+q^2x=0 \qquad (28)$$

analogous with (2), although differing from it by a factor q^2. (q does not need to be a whole number in free vibrations). It is remarked, that the approximations so far made do not concern the freely variable system by (20) or (24) in which system free vibrations are given exactly by (28). This may be a guide for the judgement of cases where α-corrections will be necessary. On the other hand, equation (28) is valid also for a single vibration form in a multi-mass system to the same extent as it is valid for a one-mass system.

Solving equation (28) the next question of approximation concerns the β-value which is usually so small that β in 2. power may be disregarded in practical calculations. For $\beta = 0$, that means for a linear, constant-mass system representing the mean value of the real system, the solution would be an arbitrary sinus vibration

$$x = a\sin(q\alpha+\varphi)$$

Introducing this in (28) the products $\cos 2\alpha\cdot\frac{d^2x}{d\alpha^2}$ and $\sin 2\alpha\cdot\frac{dx}{d\alpha}$ will deliver new members of order $q-2$ and $q+2$, also being sinus terms with phase φ at the main position, and in the first β-approximation the solution must be

$$x = a\sin(q'\alpha + \varphi) + b\sin((q'-2)\alpha + \varphi) + c\sin((q'+2)\alpha + \varphi) \qquad (29)$$

in which q' might be different from q, that means, for a given ω_o the real number of vibrations pr. rev., q', might be different from the given q standing in the differential equation and meaning the number of vibrations pr. rev. in a constant-mass system.

Introducing (29) in (28) we get 3 coefficient equations for the members of order q', q'-2, and q'+2, all of them still remaining sinus terms with phase φ. From the two latter equations b and c are found, always containing a factor β. In their multiplication with $\beta\cos 2\alpha$ and $\beta\sin 2\alpha$ they will give q'-order terms (and also terms of order q'-4 and q'+4) with factor β^2, which may all be disregarded. Consequently, we have of order q' only the 2 ordinary members for the constant-mass system, so that q' = q. For an arbitrary value of a the two other equations then give us:

$$b = -k \cdot a \qquad \text{and} \qquad c = l \cdot a$$

$$\text{with} \qquad k = \frac{1}{8}\beta\,\frac{q(q-2)}{q-1} \qquad \text{and} \qquad l = \frac{1}{8}\beta\,\frac{q(q+2)}{q+1} \qquad (30)$$

and the vibration consists of a major harmonic of order q and arbitrary amplitude a, and smaller harmonics of order q-2 and q+2 and amplitudes (30) proportional to a, All of them have the same phase φ at the main position, $\alpha = 0$, but φ is itself arbitrary (and also shifting for each passagé of the main position, if q is not a whole number). As the vectors representing the 3 components are rotating at different speeds, their relative angles are continually shifting, but still they are firmly bound together by their common position when the crankshaft is at the main position, which determines the timing of the variable-mass effect.

For the 3. order component of the mass variation (with coefficient β_3 by (17)) a similar calculation is easily made. The differential equation will be similar to (28) only with 3α in stead of 2α, and also coefficient 3 in stead of 2 in the second member. There will be secondary vibration components of order q-3 and q + 3 and size:

$$b = -a \cdot \frac{1}{12} \beta_3 \frac{q(q-3)}{q-\frac{3}{2}} \qquad c = a \cdot \frac{1}{12} \beta_3 \frac{q(q+3)}{q+\frac{3}{2}} \tag{30a}$$

In the present case with a 2. order mass variation $\beta = 0.11$ and $q = 8$ we get

$$k = 0.11 \cdot \frac{6}{7} = 0.094 \qquad l = 0.11 \cdot \frac{10}{9} = 0.122$$

Fig. 8 shows the 3 harmonics and the resulting vibration during 1 revolution in the case $\varphi = 0$, all 3 components being sinus curves with 0-point at the main position I. For the sake of clearness, however, the secondary components are much exaggerated and correspond to $\beta = 0.4$. The resultant (which would be very similar for other values of φ) consists alternately of two shorter and two longer vibrations, which was also clearly seen on the torsiograms, corresponding to a "momentary frequency" (understood rather as momentary phase velocity):

$$n_m = \frac{n_e}{(1-\beta \cos 2\alpha)^{\frac{1}{2}}} \cong n_e \cdot (1 + \tfrac{1}{2}\beta \cos 2\alpha) \tag{31}$$

varying in the real case 5.5 % above and below the mean value.

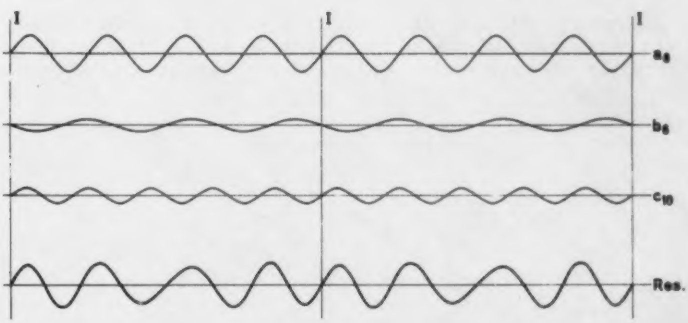


Fig. 8. Free vibration with variable frequency ($\beta = 0.40$).

It might be imagined, that eq. (31) could be used as basis for a very simple and yet correct vibration calculation, namely by expressing that the phase velocity of free vibrations ($\frac{d(\omega t)}{dt}$ in the expression $x = a \sin \omega t$) is varying as (31). In our notations we would write

$$x = a \sin q \Theta$$

where Θ is a parameter given by

$$\frac{d\Theta}{d\alpha} = 1 + \frac{1}{2}\beta \cos 2\alpha$$

The free vibration would then be (for $\varphi = 0$)

$$x = a \sin q(\alpha + \frac{1}{4}\beta \sin 2\alpha) \cong a(\sin q\alpha + \frac{1}{4}\beta q \sin 2\alpha \cdot \cos q\alpha)$$

$$= a(\sin q\alpha - \frac{1}{8}\beta q \sin(q-2)\alpha + \frac{1}{8}\beta q \sin(q+2)\alpha) \qquad (32)$$

So this simple calculation, altogether filling 3 lines, comes rather close to the solution (29) and (30) found above through lengthy operations, and in some practical applications (32) may be sufficient. But the method is correct only under the pressupposition of a high q-value, that means many vibrations pr. mass variation period, and for usual q-values the difference between the coefficients by (30) and (32) is not unimportant. Above all, we should not from this starting point be able to proceed further with the theoretical investigation as done in the following.

Strangely enough, (32) would be correct, if the Corioli's force were half as big as it is, which is easily seen from (30) where the ciffer 2 in the numerators stand for Corioli's force.

8. Fictive forces.

Subharmonic vibrations due to mass-variation

The procedure followed above in solving equation (28) is, mathematically seen, a simple iteration depending on $\beta < 1$ (as it must necessarily be in our case). The main vibration is the first approximation, the secondary vibrations the 2. approximation, and in this way we could continue in rapid convergence as β is usually $<<1$.

This procedure can be interpreted physically in a way very useful for the further investigations. When introducing the main vibration $x = a\sin(q\alpha + \varphi)$ in the differential equation the new members formed of order $q-2$ and $q+2$ (that means the members not belonging to the linear parts of the differential equation) can be considered as "fictive forces" in the distinct meaning, that when we add these forces (and their possible later derivations in the following approximations) on the system as outer impulses, the system is to be calculated as a linear system with constant swing masses equal to their real mean values. It is in the same way that the name "fictive forces" is used in relative movement. They are in fact very real forces acting on the given, real system, but "fictive" when we apply them on a simplified modification of the system in such a way as to make the result correct in the real system.

In our case the main vibration or any forced vibration (calculated in the system as linear) is a first approximation for the solution, and its fictive forces represent, in their action on the system as linear, the next and usually sufficient approximation. Looking at the matter in this way we get a clear understanding of the various vibration phenomena, and also a great simplification of all detail calculations, partly because the simple expression for a fictive force is a kind of "operator" to be used many times in actual calculations, partly because these operators are vectors of given orders and phase angles which are easy to handle, and we get all the advantages of vector calculation. Especially for multi-

mass systems (and all vibration systems of real practical importance are multi-mass systems) it would seem impossible to get through the calculations in any way than this.

Dividing equation (28) with q^2 its 3 members - an inertia force, a Corioli's force and an elastic force - will be expressed in terms of "static amplitudes". We wish to consider the new members, the fictive forces, as outer impulses and must therefore transfer them to the right side, changing their sign. From the q-order main vibration we get then, by the few members of order q-2, a fictive force of this order, expressed as static amplitude:

$$\left.\begin{array}{l} \dot{a}'_{q-2} = -a \cdot \frac{1}{2}\beta \frac{q^2-2q}{q^2} = -a \cdot \frac{1}{2}\beta \frac{q-2}{q} \\ \\ \text{and a fictive force of order q+2:} \\ \\ \dot{a}'_{q+2} = -a \cdot \frac{1}{2}\beta \frac{q^2+2q}{q^2} = -a \cdot \frac{1}{2}\beta \frac{q+2}{q} \end{array}\right\} \quad (33)$$

The fictive forces shall here always be denoted by a dot over the letter denoting the vibration which has produced them, and with index denoting the order of the fictive force itself. (33) is valid not alone for a one-mass system but also for a single vibration form in a multi-mass system with the appropriate value of β, because the action of the fictive forces from the different cylinders back upon the same vibration form is depending on the same kind of vector summation (13) as is also used in calculating β by (17). (Further about this and also about their action on other vibration forms, see section 10).

The fictive forces (33) are both sinus components with the same phase angle φ at the main position as the main vibration a which has produced them. They will themselves produce ordinary forced vibrations in the system, now considered as linear. But free vibrations mean, that the q-order main vibration is at resonance, consequently $\dot{a}'_{q-2}$ is at a relative frequency $\xi = \frac{q-2}{q}$ giving a magnification factor

$$F = \frac{1}{1-\left(\frac{q-2}{q}\right)^2} = \frac{q^2}{4(q-1)}$$

and a forced vibration

$$b = F \cdot a'_{q-2} = -a \cdot \frac{1}{8}\beta\,\frac{q(q-2)}{q-1}$$

in accordance with (30). Similary we get for the q+2 order vibration:

$$c = -a \cdot \frac{1}{2}\beta\,\frac{q+2}{q} \cdot \left(-\frac{q^2}{4(q+1)}\right) = a \cdot \frac{1}{8}\beta\,\frac{q(q+2)}{q+1}$$

Thus the secondary components of the free vibration are in reality forced vibrations coming from the fictive forces which are produced by the main vibration. Therefore the secondary components have also the same phase as the fictive forces (and as the main vibration in the main position when $q > 2$), and they have different sign because one is forced in under-resonance and the other in over-resonance.

Fictive forces are not produced exclusively by resonance vibrations, but by any forced vibration which may exist in the system. F. inst., if we have a forced vibration of order q-2, $x = b\sin(q-2)\alpha$, caused by an impulse of this order, we find by introduction in (28) a fictive force of order q

$$\left.\begin{aligned} b'_q &= -b \cdot \frac{1}{2}\beta\,\frac{(q-2)^2 + 2(q-2)}{q^2} = -b \cdot \frac{1}{2}\beta\,\frac{q-2}{q} \\ &\text{and similarly from a c-vibration of order } q+2: \\ c'_q &= -c \cdot \frac{1}{2}\beta\,\frac{(q+2)^2 - 2(q+2)}{q^2} = -c \cdot \frac{1}{2}\beta\,\frac{q+2}{q} \end{aligned}\right\} \quad (34)$$

accordingly the same "coupling factors" as by (33) from a q-order (free or forced) vibration to its accompanying fictive forces of order q-2 and q+2. It is very satisfactory that we have thus the same coupling factor going both ways, but this would not have been the case if we had not got the member expressing Corioli's force in the differential equation.

If b and c in (34) are themselves just the secondary components caused by the main component $a\sin(q\alpha + \varphi)$ of a free vibration through its fictive forces (33), the secondary fictive forces (34) will

be a kind of "reflection" back upon the free vibration. They will be not alone of the same order but also the same phase angle as this, and they will therefore act like elastic forces, changing the resonance frequency a little. Their size is together

$$b'_q + c'_q = a \cdot \frac{1}{8}\beta \frac{q(q-2)}{q-1} \cdot \frac{1}{2}\beta \frac{q-2}{q} - a \cdot \frac{1}{8}\beta \frac{q(q+2)}{q+1} \cdot \frac{1}{2}\beta \frac{q+2}{q}$$

$$= -a \cdot \frac{1}{16}\beta^2 \left[\frac{(q+2)^2}{q+1} - \frac{(q-2)^2}{q-1} \right] = -a \cdot \frac{1}{8}\beta^2 \frac{3q^2-4}{q^2-1} \cong -a \cdot \frac{3}{8}\beta^2 \qquad (35)$$

accordingly an outer impulse opposed to the vibration amplitude, therefore adding to the elastic force and increasing the frequency. Or we can say, that the order number of free vibrations is indeed increased to q' as anticipated in (29), but the difference depends on β in 2. power and was therefore disregarded in first approximation. If we now introduce (35) as outer impulse in the differential equation (28) linearized, we get

$$a \cdot \left(1 - \frac{q'^2}{q^2}\right) = -a \cdot \frac{3}{8}\beta^2 \qquad q'^2 = q^2\left(1 + \frac{3}{8}\beta^2\right)$$

and the mean frequency of free vibrations will be

$$n'_e \cong n_e \cdot \left(1 + \frac{3}{16}\beta^2\right) \qquad (36)$$

The same result is found from (31) if we add the next member in the series

$$\frac{n_e}{(1-\beta\cos 2\alpha)^{\frac{1}{2}}} \cong n_e\left(1+\frac{1}{2}\beta\cos 2\alpha+\frac{3}{8}\beta^2\cos^2 2\alpha\right) = n_e\left(1+\frac{1}{2}\beta\cos 2\alpha+\frac{3}{16}\beta^2(1+\cos 4\alpha)\right)$$

giving a mean value (36). Also this calculation presupposes a rather high q-value, otherwise the effect is not fully developed, and as seen from (35) the correction is in reality sligthly less than given by the last figure.

In the present case with $\beta = 0.11$ the frequency increase is only 0.2 % and without practical importance, but it is of some interest to note that also in the variable-mass system the mean frequencies of free vibrations in the various vibration forms are absolutely constant,

not changing with the size of the vibrations as in other kinds of (true) non-linear systems. (But the introduction of fictive forces is just as useful in these cases, see end of section 9).

For the other components of the mass variation we shall, of course, get fictive forces similar to those calculated above and acting as additional outer impulses. It follows, that the different components are in no way interfering with each other and can be calculated separately as the case may be.

By means of fictive forces we can easily solve the problem of subharmonic vibrations due to the variable-mass effect, which is the sole subject of the before mentioned highly theoretical papers (litt. 1 - 4), although the authors do not clearly mention that it is in fact subharmonic vibrations they calculate. They only speak about a "splitting-up" of the resonance points in 2 neighbouring frequencies at which steady, free vibrations can exist, and between them a range of "instability", as already described in sect. 3 in connection with subharmonic resonance vibrations exited by a cylinder impulse of the double frequency, but they do not clearly state, that in the present case (where cylinder impulses are not necessary) the vibrations are directly exited by the mass variation of the double frequency through a fictive force produced by the vibration itself. Therefore the energy supply to the vibrations remains a mystery to them, because, how can instable vibrations arise and increase without impulses supplying the energy? Of course the mass variation is always in rotating systems accompanied by an outer impulse, but it is here of double frequency of the vibrations and cannot supply energy to them (except through impulse distortion which these authors do not consider, and besides we can have subharmonic excitation even if the total outer impulse is 0).

We shall here first treat the matter for a freely variable system for which the free vibrations are exactly given by eq. (28) with α as a pure time variable (whether the system is rotating or not does not matter, because the outer impulse from the mass variation will here also be a pure time function, not subject to impulse distortion). We can without trouble generalize eq. (28) to be valid for any of the 3 mass-variation components of order p = 1-3, namely in the form:

$$(1-\beta_p \cos p\alpha)\cdot\frac{d^2x}{d\alpha^2}+p\beta_p \sin p\alpha\cdot\frac{dx}{d\alpha}+q^2x=0 \tag{37}$$

The subharmonic condition is $p = 2q$. Introducing $x = a\sin(q\alpha+\varphi)$ we get a fictive force, which expressed in full will be:

$$a'_q = a \cdot (\tfrac{1}{2}\beta_p - \beta_p) \sin(q\alpha - \varphi) = -a \cdot \tfrac{1}{2}\beta_p \sin(q\alpha - \varphi) \tag{10 a}$$

This force is in every respect analogous with (10). It is in resonance with the vibration, and the vibration is (due to the conversion of the sign of φ) free to adjust itself to any suitable phase difference. If there is damping, we have the limit condition

$$\rho = \tfrac{1}{2}\beta_p \tag{11 a}$$

and if $\rho < \tfrac{1}{2}\beta_p$ the range of subharmonic vibrations is given as

$$\xi \cong 1 \pm \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\beta_p)^2 - \rho^2} \tag{12 a}$$

For $\rho = 0$ we have $\xi \cong 1 \pm \tfrac{1}{4}\beta_p$ which is the main result of the said papers (litt. 1-4). In practice this result is quite insufficient, partly because we have always a damping, and partly because we have always a big cylinder impulse of the same order as the mass variation and subject to impulse distortion, producing the artificial impulse (10). Of course there is great difference between "artificial impulses" due to impulse distortion and "fictive forces" due to mass variation, and we shall here retain these different names, but there is also a great similarity between them, particularly the feature that their size and phase in multi-cylinder engines are depending on the same kind of vector summation, namely a_u by (13) and β_p by (17). In the formulae we can therefore without mistake use the same kind of notation for them, as they are in any case to be added directly if they are of the same order.

Thus, in calculating subharmonic vibrations in crankshafts we shall add the artificial impulse (10), $-q\, a_{pu} \cdot a$, and the fictive force (10a), $-\tfrac{1}{2}\beta_p \cdot a$, if they are both of a size to be considered. a_{pu} may exist for any order number p, while β_p is here considered only for p = 1-3. But in the cases where the mass variation is considerable we have at the same time a resulting oscillating-mass impulse, also subject to impulse distortion. It should therefore be included in a_{pu}, what it also is if we in calculating a_{pu} by (13) already include the oscillating-mass impulse pr. cyl. in the cylinder impulse. As there is a certain resemblance between the two effects of the variable masses (one through the inertia- and Corioli's force, and the other through impulse distortion) we need clear distinctions, and we shall here provide that a_{pu} always means the resultant of cylinder impulses only, while we shall use the notation a^*_{pu} for the resultant of cylinder impulses and oscillating-mass impulses. These latter are given by (16) for a one-cylinder engine, and the expression is valid also for multi-cylinder engines with β_p in stead of β_{pc}. Furthermore,

the phase difference between cylinder impulse and oscillating-mass impulse is negligible, so that we have generally:

$$a^*_{pu} = a_{pu} - \frac{p}{2q^2}\,\beta_p \tag{16 a}$$

In the subharmonic case $p = 2q$ we have

$$a^*_{pu} = a_{pu} - \frac{1}{q}\,\beta_p$$

and the total fictive force will be

$$a'_q = -a\cdot(qa^*_{pu} + \frac{1}{2}\beta_p) = -a\cdot(qa_{pu} - \frac{1}{2}\beta_p) \tag{10b}$$

that means, the force is the same as would be calculated from the cylinder impulses K_p after adding only half of the oscillating-mass impulse, a curious but quite satisfactory result. We have now for the limit damping:

$$\rho = \left| qa_{pu} - \frac{1}{2}\beta_p \right| \tag{11 b}$$

specially for $p = 2$ $\qquad \rho = \left| a_{2u} - \frac{1}{2}\beta \right|$

and for the range of the vibrations:

$$\xi \cong 1 \pm \frac{1}{2}\sqrt{(qa_{pu} - \frac{1}{2}\beta_p)^2 - \rho^2} \tag{12 b}$$

This formula includes all possibilities $p = 2q$ and combines our results from sect. 3 with the results from the previous papers on the subject (litt. 1-4), which do not consider impulse distortion and damping, but agree on the member $\frac{1}{2}\beta_p$. In litt. 5 (from a much later date) due regard is taken to damping but not to impulse distortion.

Regarding the question of energy supply it is now obvious, that in the freely variable system (whether rotating or not) the energy must be supplied by the driving mechanism, for the moving mass. In these cases the fictive force is $-a\cdot\frac{1}{2}\beta_p$, and at resonance it performs a work pr. q vibrations

$$A = \frac{1}{2}\,q\pi\beta_p\,a^2\cdot m\omega_e^2$$

The driving mechanism is pr. p periods performing a work against the centrifugal force (21):

$$A = -\int_{p\sim} C \cdot \frac{dy}{dt}dt = -G_oR \cdot \int \left(\frac{dx}{dt}\right)^2 \cdot \frac{dy}{dt}dt$$

where $\quad y = -f\cos p\alpha \qquad \frac{dy}{dt} = f p\omega_o \sin p\alpha$

$$x = a\sin(q\alpha + \frac{\pi}{4}) \qquad \left(\frac{dx}{dt}\right)^2 = a^2q^2\omega_o{}^2 \cdot \frac{1}{2}(1 + \cos(p\alpha + \frac{\pi}{2})) = \frac{1}{2}a^2\omega_e^2(1 - \sin p\alpha)$$

and $\quad 2\,G_oRf = \beta_p m \qquad$ giving

$$A = -\frac{1}{2}pG_oRfa^2\omega_e^2 \int_0^{2\pi} \sin p\alpha \cdot (1 - \sin p\alpha)d\alpha = \frac{1}{2}q\pi\beta_p ma^2\omega_e^2$$

For the crankshaft system the matter is more complicated. We have here the total fictive force (10b). The first part of it, $-a \cdot qa\beta_u$, is the primary artificial impulse which, as shown in sect. 3, is "braked" by its accompanying secondary artificial impulse. The other part of it, $-a \cdot \frac{1}{2}\beta_p$, is the same as in the freely variable system and gives the same impulse work A. Braking work to the same amount must then be supplied by the members in the correct differential equation for free vibrations (25), which are not effective in the freely variable system. We shall, however, leave out the variable-mass impulse which has been included in $a\beta_u$. In generalized form eq. (25) is then:

$$(1-\beta_p\cos p\alpha) \cdot \frac{d^2x}{d\alpha^2} + p\beta_p\sin p\alpha \cdot \frac{dx}{d\alpha} + \frac{1}{2}p\beta_p\sin p\alpha \cdot \left(\frac{dx}{d\alpha}\right)^2 + q^2x = 0 \qquad (38)$$

where α in $\cos p\alpha$ and $\sin p\alpha$ in reality means $\omega_o t + x$. Retaining the letter α as a time variable, $\cos p\alpha$ and $\sin p\alpha$ should be, respectively

$\cos p\alpha - px\sin p\alpha \qquad$ and $\qquad \sin p\alpha + px\cos p\alpha$

We have then 3 new members:

$$N = p\beta_p x\frac{d^2x}{d\alpha^2} \cdot \sin p\alpha + p^2\beta_p x\frac{dx}{d\alpha} \cdot \cos p\alpha + \frac{1}{2}p\beta_p\left(\frac{dx}{d\alpha}\right)^2 \sin p\alpha \qquad (39)$$

one from the inertia force, one from the Corioli's force, and one from the "centrifugal force", which all contain the product of two of the functions x and its derivatives and therefore will give a braking momentum proportional to a^2.

For $x = a\sin(q\alpha + \frac{\pi}{4})$ we have, similarly as above:

$$x\frac{d^2x}{d\alpha^2} = -\frac{1}{2}a^2q^2(1+\sin p\alpha), \qquad x\frac{dx}{d\alpha} = \frac{1}{2}a^2q\cos p\alpha \qquad \text{and}$$

$$(\frac{dx}{d\alpha})^2 = \frac{1}{2}a^2q^2(1-\sin p\alpha)$$

In order to get dimension of static amplitudes eq. (38) and N should be divided with q^2, and we get the braking work pr. revolution:

$$\int_0^{2\pi}\frac{N}{q^2}d\alpha\cdot m\omega_e^2 = 2\pi p\beta_p\cdot\frac{1}{2}a^2(-\frac{1}{2}+\frac{p}{2q}-\frac{1}{4})\cdot m\omega_e^2 = -2A+4A-A = A$$

The "centrifugal force" gives here a braking work -A, that means a forward pull to this amount, but the inertia- and Corioli's forces give together a braking work 2A. It follows, of course, from the energy principle, that in crankshaft vibrations any kind of impulse work, whether from real or fictive impulses, must be counterbalanced by a braking work, and there is nothing surprising in this, but we have found it advisable in this case to show the calculation in details as a final control on the above developed formulae, especially formula (30) (and the corresponding (41) in next section) which are fundamental for practical calculations. Whether such expressions are a little more or less complicated (f. inst. whether the coefficients should be taken by (30) or by (32)) does not mean anything in the calculation work, but it is very satisfactory - when we have to be content with a 1. degree approximation - that this approximation is at least completely correct within its own limits.

We can now also make up for the member (26) which we inadvertently disregarded in the calculation of free vibrations, sect. 7, and which means the impulse distortion of the oscillating-mass impulse. But in real engines, even in "free" vibrations, we cannot have such a member without also having distortion of the cylinder impulse of the same order. The 2 impulses together are given in (16 a), and they produce a fictive force $a\cdot\frac{1}{2}pa^*_{pu}$ which should be added to the main fictive forces (33). Expressing these in a more general form we finally get:

$$a'_{q\pm p} = -a\cdot(\frac{1}{2}\beta_p\frac{q+p}{q} - \frac{1}{2}pa^*_{pu}) \qquad (33\,a)$$

The expression is valid for all values of q and p. Only for p = 1-3 we have a mass-variation effect, and for $p > 3$ the first member in the parenthesis disappears, but the other member remains as $a \cdot \frac{1}{2} p a_{pu}$ which is the usual impulse distortion of a p-order impulse by a q-order vibration. For p = 2q the lower fictive force (33a) reduces to the subharmonic value (10b), although with opposite sign because (33a) gives $a'_{-q} = -a'_q$. In the subharmonic case we need the correct expression (10b) because the two forces will probably be of the same order of size, as a_{pu} will be big due to the extraordinarily low frequency in such cases. But in normal engines running at much higher q-values a^*_{pu} will usually be small in comparison with β_p, and in the general theory we shall therefore use expression (33) or generalized as the first member in (33a).

(33a) shows in any case, that variable-mass effect and impulse distortion are two things which can be considered separately - except in the subharmonic case, where their combined effect is decisive for the existence or non-existence of the vibrations.

9. Secondary resonance.

The use of fictive forces in true non-linear systems

It is obvious, that "non-linear" vibrations of the kind (29) and fig. 8, having in their free state secondary components of order $q-2$ and $q+2$, can be excited by impulses of one of these orders, when the main vibration of order q is at or near resonance. All three components are present in the free vibration, and when impulse K_q is in resonance with the main component, impulses K_{q-2} and K_{q+2} can be said to be in "secondary resonance" with the vibration, as they are obviously able to deliver impulse work on the total vibration although in much smaller degree than the impulse K_q in "direct resonance".

The case most likely to be met with in practice is when the impulses K_q in a multi-cylinder engine give a very small resulting static amplitude a_{qs}, so that this resonance point would be considered unimportant in the usual calculations, while the impulses K_{q-2} give a very big $a_{(q-2)s}$, the direct resonance of which would be situated well above maximum revolutions and therefore also disregarded in usual calculations. If in such a case the system has also a considerable 2. order resulting mass variation β, the vibration can be excited to a rather big size by the impulse K_{q-2} in secondary resonance. This was just the case mentioned in the introduction and calculated for mass variation in section 4 and for free vibrations in section 7.

The excitation in secondary resonance is very easy to calculate by means of the theory of fictive forces. The impulse K_{q-2} with static amplitude $a_{(q-2)s}$ will at the resonance point $\omega_o = \frac{\omega_e}{q}$ give a forced vibration of order $q-2$ and amplitude:

$$b_1 = a_{(q-2)s} \cdot \frac{q^2}{4(q-1)} \tag{40}$$

which according to (34) gives a fictive force of order q:

$$b'_q = -b_1 \cdot \frac{1}{2}\beta\,\frac{q-2}{q} = -a_{(q-2)s} \cdot \frac{1}{8}\beta\,\frac{q(q-2)}{q-1} = -k \cdot a_{(q-2)s} \tag{41}$$

i.e. the same transformation factor, $-k$, as between the components of the free vibration given by (30). The main vibration is then given as

$$a = b'_q \cdot F = \frac{k \cdot a_{(q-2)s}}{\sqrt{(1-\xi^2)^2 + \rho^2\xi^2}} \tag{42}$$

with suitable phase difference between a and b'_q, as the fictive force is acting completely like an ordinary impulse, and the vibration is free to adjust itself to the right phase difference. Therefore the whole stress curve about the resonance point is of the same shape as in linear vibrations.

The matter seems very simple, especially as the transformation factor (41) is equal to the relation between the components of the free vibration, and yet it would not be quite correct to think, that the impulse K_{q-2} is acting directly on the secondary component of the vibration which is, after all, a rather loose and just "secondary" apparation. The real thing is, that the impulse K_{q-2} is in the first place producing a very definite and constant forced vibration of its own order, and it can then be imagined (as the calculations also show), that this forced vibration through the inner mechanism of the system can produce exciting forces of other orders. The secondary component of the free vibration is only an outward sign, that such a mechanism exists, which will also work the other way.

Therefore the forced vibration b_1 has nothing to do with the b-component of the excited vibration. b_1 is fixed at the same phase as the resulting impulse $a_{(q-2)s}$, while the b-component changes phase together with the main vibration as the degree of resonance may require. It is then so, that the exciting force b'_q is depending solely on b_1 which is constant in the interval to be considered (or only slightly increasing with the revolutions) and does not require any impulse work. But at resonance, when the main vibration a follows in phase $\frac{\pi}{2}$ after b'_q, the b-component will follow $\frac{\pi}{2}$ after $a_{(q-2)s}$, and the impulse can then deliver work on it

(although not the full impulse work consumed by the vibrations, as shown below).

Usually the impulse K_q in direct resonance cannot be completely disregarded even if its resulting static amplitude a_{qs} is comparatively small. a_{qs} should be added geometrically to b'_q, and this is easily done by considering the crankshaft in main position. Here the fictive force is always in phase opposite to the forced vibration which has produced it. In practical calculations it will be more convenient to use the ordinary vector diagrams for impulse summation drawn for crank 1 in top. Then, if μ is the angle by which the crankshaft in that position has passed the main position, b'_q will be at an angle 2μ past the opposite of $a_{(q-2)s}$ (see also fig. 10).

When using the corrected value for the coupling factor (33 a), which is also applicable to (34) because the coupling is the same both ways, and further introducing a general expression for the forced vibration b_1 we get a complete formula for fictive forces of order q, namely

$$b'_q \text{ or } c'_q = a_{(q \pm p)s} \cdot \frac{q^2}{p(p \pm 2q)} \cdot \left(\frac{1}{2}\beta_p \frac{q \mp p}{q} - \frac{1}{2} p a^*_{pu}\right) \qquad (41\,a)$$

From this it is seen, that a fictive exciting force for a q-order resonance vibration may be produced by any 2 impulses for which the sum or the difference of the order numbers is equal to q. The conditions are, that one of the impulses should give a big a_s-summation and consequently a comparatively big forced vibration, while the other impulse should give a big a_u-summation, consequently a considerable impulse distortion or mass variation to be worked upon by the forced vibration.

Most of these cases are, however, subtleties, and in practice we shall probably never meet this kind of excitation of any important size except in the case $p = 2$ and impulse of order $q-2$. Even in that case the conditions are narrow, namely that both β and $a_{(q-2)s}$ should be big, and the effect depends on the product of them.

But then we have also the subharmonic cases $p = 2q$ which, seen from this point of view, are curious singularities at which the resonance vibration itself plays the role of the otherwise necessary forced vibration, although with negative sign, namely for

$$q - p = -q$$

When the vibration comes up at all, its disturbing effect is much bigger than that of an ordinary forced vibration, and there is now only one condition to be fulfilled, namely that a_{pu} should be big (correctly $qa_{pu} - \frac{1}{2}\beta_p$). This requires a rather low p-value, but not necessarily as low as 2. We shall therefore not give any definite rule for these cases but only say, that any case $p = 2q$ at a comparatively low p-value should be regarded with suspicion, untill it has been calculated at least approximately.

We shall finally investigate the energy balance in the case of excitation by an impulse K_{q-2}. Assuming $\gamma = 0$ (its value is without influence on the size of the vibrations) and writing a_s short for $a_{(q-2)s}$ we have at sharp resonance:

Impulse	$a_s \cdot \sin(q-2)\alpha$	
Forced vibration	$b_1 \cdot \sin(q-2)\alpha$	with $b_1 = a_s \cdot \frac{q^2}{4(q-1)}$
Fictive force	$- k a_s \cdot \sin\alpha$	with $k = \frac{1}{8}\beta\frac{q(q-2)}{q-1}$
Main vibration	$a \cdot \cos q\alpha$	
b-component	$- k a \cdot \cos(q-2)\alpha$	

The work done by the fictive force on the main vibration is pr. revolution (in relative measure):

$$q\pi k a a_s$$

while the work done by the real impulse on the b-component is only

$$(q-2)\pi k a a_s$$

The missing work

$$A = 2\pi k a a_s$$

is in the freely variable system delivered by the driving mechanism for the moving mass. In $(dx/dt)^2$ determining the centrifugal power (21) the double product of the velocities of the main vibration and the forced vibration will namely give a member with $\sin 2\alpha$ which by multiplication with the $\sin 2\alpha$ in dy/dt gives a constant of the right size.

In the crankshaft system we have the 3 "braking members" (39), for p = 2:

$$N = 2\beta x\frac{d^2x}{d\alpha^2}\sin 2\alpha + 4\beta x\frac{dx}{d\alpha}\cos 2\alpha + \beta\left(\frac{dx}{d\alpha}\right)^2\sin 2\alpha$$

Here the products of the main vibration and the forced vibration give us the following members of 2. order:

from $x\frac{d^2x}{d\alpha^2}$: $\frac{1}{2}ab_1\left((q-2)^2 + q^2\right)\sin 2\alpha = ab_1\left(q^2-2q+2\right)\sin 2\alpha$

from $x\frac{dx}{d\alpha}$: $\frac{1}{2}ab_1\,(q-2-q)\cos 2\alpha = -ab_1\cos 2\alpha$

from $\left(\frac{dx}{d\alpha}\right)$: $-ab_1\,q(q-2)\sin 2\alpha$

The braking work $\int_0^{2\pi}\frac{N}{q^2}\,d\alpha$ is now from the inertia- and Corioli's force together:

$$2\pi\beta\,\frac{ab_1}{q^2}\left(q^2-2q+2-2\right) = 2\pi a\beta a_s\cdot\frac{q(q-2)}{4(q-1)} = 4\pi\,k\,a\,a_s = 2A$$

and from the "centrifugal force":

$$-2\pi\beta\,\frac{ab_1}{q^2}\,q(q-2)\cdot\frac{1}{2} = -2\pi\,k\,a\,a_s = -A$$

so that we have the same as in the subharmonic case, sect. 8: The centrifugal force gives a forward pull, or braking work -A, while the two other forces together give a braking work 2 A. The total braking work A from the 3 members is proportional to a_s because it is proportional to b_1, and it is transferred to impulse work without the direct action of a_s. But a_s is delivering the rest of the impulse work, thereby consuming a braking work to the same amount through usual impulse distortion.

In true non-linear systems in which the variable coefficients of the differential equation are depending solely on the vibration amplitude the use of fictive forces will give the same simplification of the calculations as in the freely variable (in mathematical sense "linear") and the guided variable systems. We shall here just show the application of fictive forces on the simplest possible non-linear system, namely a one-mass system with variable stiffness

$$s = s_o(1 + \delta \cdot x) \qquad \text{where} \quad |\delta \cdot x| \ll 1$$

Using $\alpha = \omega_e t$ as independent variable with $\omega_e^2 = \frac{s_o}{m}$ the differential equation for free vibrations is:

$$\frac{d^2x}{d\alpha^2} + x(1 + \delta x) = 0 \qquad \text{or} \qquad x'' + x = -\delta \cdot x^2$$

so that the fictive force is simply $-\delta x^2$.

For the free main vibration $x = a\sin\alpha$ we have fictive forces of order 0 and 2, namely

$$a_o' = -\frac{1}{2}\delta a^2 \qquad \text{and} \qquad a_2' = \frac{1}{2}\delta a^2 \cdot \cos 2\alpha$$

consequently there is a 2. order component in the free vibration besides a constant displacement of less practical importance. Secondary resonance and subharmonic excitation is here the same thing, and it would be impossible to say which name is the more correct. In the freely variable system secondary resonance means, that the exciting force is produced by a forced vibration, while subharmonic excitation means, that it is produced by the main vibration itself, but in the true non-linear system the exciting force is produced by the product of the main vibration and a forced vibration. This latter, forced by a 2. order impulse a_{2s}, is

$$b_1 = \frac{1}{1 - 2^2} \cdot a_{2s} = -\frac{1}{3} a_{2s}$$

and the total vibration is (disregarding the constant displacement):

$$x = a\sin(\alpha + \varphi) + b_1 \sin 2\alpha$$

giving a 1. order fictive force

$$b_1' = -\delta \cdot 2ab_1 \cdot \frac{1}{2} = a \cdot \frac{1}{3} \delta \cdot a_{2s} \qquad (10\,c)$$

with free phase difference in relation to a, so that the equilibrium condition in the resonance case is

$$\rho = \frac{1}{3} \delta a_{2s} \qquad (11\,c)$$

It will not be possible to obtain this result in an easier way. So in this method we cannot distinguish between freely variable and true non-linear systems, because in both cases the fictive forces represent everything besides the "linear" forces coming from the constant members in the differential equation. But the fictive forces can be of very different character in the different cases.

The fictive force (10 c) has the character of subharmonic excitation as it is proportional to a like the force (10 a) in a freely variable system, but it has also the character of secondary resonance, as it is proportional to an a_s like the force (41). Therefore, the presence of a forced vibration and the absence of damping above a certain value (11 c) in relation to this are the conditions for the vibrations, and although the exciting force itself is only appearing at the same time as the vibration, this will inevitably arise at the slightest, unavoidable disturbance.

In the case $s = s_o(1 + \varepsilon \cdot x^2)$, for which the differential equation is called <u>Duffing's</u> equation, we get similarly:

$$b_1' = a^2 \cdot \frac{3}{32} \varepsilon \cdot a_{3s} \qquad \text{and} \qquad \rho = a \cdot \frac{3}{32} \varepsilon \cdot a_{3s} \qquad (11\,d)$$

for subharmonic vibrations caused by a 3. order impulse a_{3s}.

In this case a given (small) damping can prevent small vibrations but not bigger. The vibrations will not come up by themselves, but they can be started by a single push, and they will then increase rapidly.

All of the vibration energy is in these cases supplied directly by the impulse, as there is no special mechanism involved in the variation.

In <u>relaxation vibrations</u> following <u>van der Pol's</u> equation

$$x'' - \varepsilon(1 - x^2) \cdot x' + x = 0$$

we have a total fictive force $\varepsilon(1-x^2)x'$. Introducing $x = a\sin(\alpha+\varphi)$ we get a 1. order fictive force

$$\varepsilon\, a\cos(\alpha+\varphi)\cdot\left(1 - a^2\cdot\frac{1-\cos 2(\alpha+\varphi)}{2}\right)$$

$$= \varepsilon\, a\cos(\alpha+\varphi)\cdot\left(1-\frac{a^2}{2}\right) + \varepsilon\, a\frac{a^2}{2}\cos(\alpha+\varphi)\cdot\cos 2(\alpha+\varphi)$$

$$\cong \varepsilon\, a\cos(\alpha+\varphi)\cdot\left(1-\frac{a^2}{2}\right) + \varepsilon\, a\cos(\alpha+\varphi)\frac{a^2}{4} = \varepsilon\, a\left(1-\frac{a^2}{4}\right)\cdot\cos(\alpha+\varphi)$$

The vibration is then creating its own impulse through some special mechanism, also supplying the energy from outside, just as in the case of subharmonics in the crankshaft system. This artificial impulse is of constant phase difference, for $a < 2$ always leading the vibration by 1/4 period and causing increasing vibrations until $a = 2$, at which point the 1. order impulse is 0, and steady, free vibrations can exist, although distorted by secondary components of forced vibrations caused by the fictive forces of higher orders. For big values of ε these distortions will be big and also reduce the frequency of free vibrations, but the maximum amplitude remains at $a = 2$. (see litt. 11, p. 152 f).

Also for calculation of true non-linear multi-mass systems the method of fictive forces will be useful. We shall not treat the matter here but only say, that in stead of the summation $\Sigma\, m\,\Delta^2$ representing kinetic energy comes here the summation $\Sigma\, s\cdot d^2$ representing potential energy, s meaning the stiffness of the various parts of the system and d their difference amplitudes. By this it is easy to determine an equivalent one-mass system for each vibration form, although the matter is quite different from (or nearly opposite to) the problems of freely variable systems, as the fictive forces in the true non-linear system are not single, free forces but difference forces between neighbouring masses. Therefore we must here in stead of the usual rope polygon fig. 3, representing amplitudes over the elasticities as abscissae, use the complementary rope polygon

representing forces over the masses as abscissae. The fictive "forces" (force differences) will all be in same phase, so that we have no trouble with phase angles in this case. On the other hand, they will not be proportional to the "vibration form" (force form) which has produced them, but rather to some higher power of it, and they will therefore contain components also in the other vibration forms causing coupling effects similar to those treated in the next section.

10. Coupled vibrations in different vibration forms.

Coupled subharmonic vibrations.

The vector diagram fig. 5 can now be considered as representing fictive forces of order $q-2$ or $q+2$ in their action back upon the vibration form of the q-order vibration which has produced them. The full understanding of this requires, however, a little reflection.

The fictive forces from each individual cylinder has not, like ordinary cylinder impulses, a given phase when the crank for that cylinder is in top position, but they have in this position the <u>same</u> phase as the vibration which produces them (correctly: they are in phase opposite to the vibration as they have opposite sign). From the time when a crank is in top until the time for which the vector diagram is drawn the crank has turned an angle θ, the vibration vector has turned an angle $q\theta$, and the fictive force vector an angle $(q \pm 2)\theta$, the difference being $\pm 2\theta$. At that time the vibration vector is in some phase which in any case is common for all cylinders, and the diagram gives therefore the correct relative position of the fictive force vectors. Their size is proportional to $\beta_c \Delta_a$, and as the vectors are set up in the diagram as $\beta_c \Delta_a^2$, they represent impulse work vectors, and the resultant R represents the resulting fictive force expressed in terms of static amplitude as calculated in the formulae above.

There is now only one thing more we need to know, namely the real phase angle of the resulting fictive force at a given time, but as fictive forces and mass variations are composed by the same vector diagram, the above rule for a single cylinder can be extended to the whole vibration form, so that the rule is: When the crankshaft is in the <u>main position</u>, the resulting fictive force vector is always opposite to the vector of the vibration which has produced it. Although the two vectors are rotating at different speeds, they are timed together at the main position, at which resultant R, fig. 5, has phase 0. This means, that if the vibration

has here the phase 0, the resulting fictive force has phase π, and if the vibration has another phase, the fictive force is always opposite to it. The argument is valid both for fictive forces of order q-2 and q+2.

So far there is nothing new in this, but it is now seen, that we can easily calculate the action of this set of fictive forces on another vibration form B, namely by using the same vector diagram fig. 5, only with vectors of size $\beta_c \Delta_a \Delta_b$ in stead of $\beta_c \Delta_a^2$. We shall then get a coefficient of "coupled mass variation", analogous with (17) but here considered only for the 2. order mass variation:

$$\beta_{ab} = \frac{\Sigma(\beta_c m_c \Delta_a \Delta_b)}{\Sigma m \Delta_b^2} \tag{43}$$

from which we can calculate the fictive force $a'_{(q \pm 2)B}$ in vibration form B produced by vibration a_A in vibration form A, analogous with (33):

$$a'_{(q \pm 2)B} = - a_A \cdot \frac{1}{2} \beta_{ab} \frac{q \pm 2}{q} \cdot \frac{\omega_a^2}{\omega_b^2} \tag{44}$$

The notation is a little troublesome, but the idea is simple enough. β_{ab} by (43) means the action from vibration form A to vibration form B, and we have therefore put $\Sigma m \Delta_b^2$ in the denominator, but in order to get the fictive force (44) expressed in terms of static amplitude in that vibration form we must further multiply with the square of the relation between the frequencies of the two vibration forms.

In practice it is more the opposite way which is of interest, as we wish to know all fictive forces of order q in the vibration form A, whether coming from this or from other vibration forms. We need here the mass variation coefficient "from B to A":

$$\beta_{ba} = \frac{\Sigma(\beta_c m_c \Delta_a \Delta_b)}{\Sigma m \Delta_a^2} \tag{45}$$

which will give an additional fictive force, similar to (41):

$$d'_q = d_1 \cdot \frac{1}{2}\beta_{ba} \cdot \frac{q \pm 2}{q} \tag{46}$$

where d_1 means the real, forced vibration of order $q \pm 2$ calculated separately in vibration form B and therefore not to be expressed by a general formula like (40). $a_{(q \pm 2)s}$ will be smaller in the higher vibration form and the magnification factor closer to unity. The phase of d'_q is found by the usual rule, opposite to the forced vibration at the main position which, however, will be different from the main position for β (or β_{aa} as it should now be called).

Such coupled vibrations will probably never be of importance in crankshaft vibrations except in very special cases, but it is one of our purposes here to anticipate all special possibilities. As for coupled vibrations these possibilities are easily discovered by drawing the simple vector diagrams for the β-values (45). They could also be found by calculating the exact vibration form for various positions of the shaft and then note the variation in the shape of the vibration form, but this would, of course, be very troublesome.

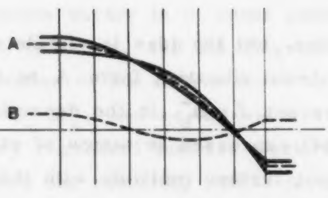


Fig. 9. The working of coupled vibrations.

Fig. 9 shows the working of the coupling in a case where there is a high β_{ba}. The full drawn curves are the real vibration forms of free vibrations, when the crankshaft is (stationary) at the main position for β_{ba} and at a position 90° from this. During rotation there will be a continuus shift between the two modifications of the vibration form, and the free vibration will contain a q-2 order component of vibration form B, which can be considered (and calculated) as a forced vibration produced by the force (44). Oppositely, a q-2 order impulse giving a big a_s in vibration form B would be in secondary resonance with the q-order vibration in A. β_{aa} may even be 0, and we have then no frequency variation, only alternately increasing and decreasing vibrations in the two cylinder groups.

We can also get coupled subharmonic vibrations in two vibration forms A and B, caused by the 2. order mass variation (or the cylinder impulse K_2), namely at a speed of revolution:

$$\omega_o = \frac{\omega_a + \omega_b}{2}$$

when ω_a and ω_b correspond to the proper frequencies of the two vibration forms and these between them have a considerable β_{ab} The numbers of free vibrations pr. revolution in the two vibration forms will namely be:

$$q_a = \frac{2\omega_a}{\omega_a + \omega_b} \qquad q_b = \frac{2\omega_b}{\omega_a + \omega_b}$$

so that $$q_a + q_b = 2 \quad \text{or} \quad \begin{cases} q_a = -(q_b - 2) \\ q_b = -(q_a - 2) \end{cases}$$

which means that each of the two vibration forms will produce fictive forces in resonance with the other vibration form. Furthermore, due to the conversion of the sign of the q-values there is also a change of sign of the phase angle φ when going over from the vibration to its accompanying fictive forces, and consequently there is free possibility of phase adjustment in the neighbourhood of resonance.

When the damping is small enough we have then increasing vibrations in both vibration forms at the same time as each component will excite the other in ever increasing degree. The matter has been investigated theoretically by F. Weidenhammer (litt. 5, 1955) and some hints of it are also found in litt. 1-4, but only considered as a question of "stability", not realizing that it is a matter of coupled subharmonic vibrations. In usual engine plants the phenomenon would only occur at impossible high numbers of revolution, but theoretically it is very interesting.

None of the vibration components has an even number of vibrations pr. rev. but together they have 2 ~ / rev. At the main position the mean value of the phase angles of the two components will at resonance be $\pi/4$. If the A-vibration at this position has a phase angle φ_a , the B-vibration will have phase angle $\varphi_b = \frac{\pi}{2} - \varphi_a$. Then the resulting fictive force in the A-vibration has phase angle

$$\pi - \varphi_a = \frac{\pi}{2} + \varphi_b$$

leading the B-vibration by $\frac{\pi}{2}$, and inversely the fictive force

in the B-vibration will have phase $\pi - \varphi_b$ leading the A-vibration by $\frac{\pi}{2}$. Outside of resonance $\frac{\varphi_a + \varphi_b}{2}$ will approximate $\frac{\pi}{2}$ or 0 (resp. above and below resonance), and the action will remain completely reciprocal.

The conditions for steady vibrations at resonance and with equilibrium value of the damping will be:

$$-a \cdot \frac{1}{2}\beta_{ab} \cdot \frac{q_a - 2}{q_a} \cdot \frac{\omega_a^2}{\omega_b^2} = a \cdot \frac{1}{2}\beta_{ab}\frac{\omega_a}{\omega_b} = b \cdot \rho_b$$

and $$-b \cdot \frac{1}{2}\beta_{ba} \cdot \frac{q_a - 2}{q_a} = b \cdot \frac{1}{2}\beta_{ba}\frac{\omega_b}{\omega_a} = a \cdot \rho_a$$

which combine into

$$\rho_a \rho_b = \frac{1}{4}\beta_{ab}\beta_{ba} \tag{47}$$

Usually ρ_a and ρ_b, respectively β_{ab} and β_{ba} will be rather similar, and (47) is then analogous with (11 a), sect. 8. For a damping smaller than this the range of increasing vibrations is given, analogous with (12 a), approximately as

$$\xi \simeq 1 \pm \frac{1}{2}\sqrt{\frac{1}{4}\beta_{ab}\beta_{ba} - \rho_a \rho_b} \tag{48}$$

Litt. 5 has a very similar result found from the stability criterion through complicated matrix calculations and confined to the special case of a 4-cylinder engine. The method does not seem suitable for generalization, and when - as in the present statement - a generalization is made, matrix calculation becomes obsolete. It is, after all, nothing but vibration analysis carried out in a very obscure manner.

At resonance with equilibrium damping the damping work pr. sec. will be the same in both vibration forms, and with smaller damping the steady vibrations at the points (48) will have the same kinetic energy in the two vibration forms. The amplitudes will be approximately inversely proportional to the frequencies.

Now, as in the case of single subharmonic vibrations, we must also consider the cylinder impulse K_2 which, through impulse distortion, will have a similar effect as the 2. order mass variation. Perhaps it will be more instructive to consider the general case of a cylinder impulse K_p (of comparatively low order number p) causing coupled subharmonic vibrations at a speed

$$\omega_o = \frac{\omega_a + \omega_b}{p}$$

It would here be convenient to construct a "coupled static amplitude"

$$a_{pab} = \frac{\Sigma(K_p \Delta_a \Delta_b)}{\omega_a \omega_b \sqrt{\Sigma m \Delta_a^2 \cdot \Sigma m \Delta_b^2}}$$

depending on the same type of vector summation as β_{ab} by (43) but with a somewhat different denominator. The equilibrium condition corresponding to (47) would then be:

$$\sqrt{\rho_a \rho_b} = \frac{1}{2} p \, a_{pab} \tag{49}$$

also analogous with (11) in sect. 3. Similarly we get for the critical range

$$\xi \cong 1 \pm \frac{1}{2} \sqrt{\left(\frac{1}{2} p a_{pab}\right)^2 - \rho_a \rho_b} \tag{50}$$

analogous with (12).

In the cases p = 1-3 where we have both mass variation and impulse distortion we can use formulae (47) and (48) when in the calculation of β_{ab} and β_{ba} by (43) and (45) β_c is replaced by

$$\beta_{pc} - \frac{p K_p}{m_c \omega_a \omega_b}$$

or we can use formulae (49) and (50) when in the calculation of a_{pab} we replace K_p by

$$K_p - \frac{\beta_{pc} m_c \omega_a \omega_b}{p} = K_p + K_{po} \cdot \frac{2\omega_a \omega_b}{(\omega_a + \omega_b)^2}$$

that means adding a fraction of the oscillating-mass impulse pr. cyl. K_{po} (see formula (15), sect. 4. It is positive for p = 1, negative for p = 2-3). This fraction was in sect. 8 found = $\frac{1}{2}$ in the case of single subharmonic vibrations, corresponding to $\omega_a = \omega_b$.

Perhaps the best way is to define the coefficient of coupled mass variation as

$$\beta_{pab} = \frac{\Sigma(\beta_{pc} m_c \Delta_a \Delta_b)}{\sqrt{\Sigma m \Delta_a^2 \cdot \Sigma m \Delta_b^2}}$$

with the full value of β_{pc}, and retain a_{pab} as calculated for the cylinder impulse only. (49) will then be:

$$\sqrt{\rho_a \rho_b} = \left| \frac{1}{2} pa_{pab} - \frac{1}{2} \beta_{pab} \right|$$

analogous with (11b), sect. 8, and we get a similar extension of formula (50).

It might be expected that there would be possibility of coupled subharmonic vibrations also at the speeds

$$\omega_o = \frac{\omega_b - \omega_a}{p} \qquad (\text{with } \omega_b > \omega_a)$$

but it is not so. In these cases we have

$$q_a = \frac{p\,\omega_a}{\omega_b - \omega_a} \qquad q_b = \frac{p\,\omega_b}{\omega_b - \omega_a}$$

therefore $q_b - q_a = p$ or $q_a = q_b - p > 0$

$$q_b = q_a + p$$

There is no shift of sign in the q-values and neither in the phase angles, consequently no possibility of phase adjustment. We have therefore in these cases no excitation of subharmonic vibrations, only a "stiff coupling" working in that way, that if there are free vibrations and the damping is small the vibrations will shift periodically from one vibration form to the other, with bigger amplitude at the lower frequency. If one of the vibration components is excited by an outer impulse (which, however, would require q-values as whole numbers) the other component is also excited, and the damping coefficient will be increased in the proportion

$$1 + \frac{\left(\frac{1}{2} pa_{pab} - \frac{1}{2} \beta_{pab}\right)^2}{\rho_a \, \rho_b}$$

It would not seem improbable to find such cases of damping increase in practice, perhaps specially in the case $\omega_b = 2\omega_a$ when a big impulse of order p would be in direct resonance with vibration form A and at the same time the artificial impulse of order 2p in resonance with vibration form B. The excited vibration B would then work back as damping on vibration A.

It is now seen, that the vibration analysis for linear multi-mass systems given in section 2 is valid also for non-linear systems, namely for the system linearized and submitted to fictive forces. The only difference is, that in the non-linear system there are coupling effects between the various vibration forms, which are not present in the linear system without damping.

It also follows that even in the linear system there may be similar coupling effects due to an unequally distributed damping. We can namely consider the damping forces as fictive forces on a damping-free system. Produced f. inst. by the main vibration in vibration form A they are all in the same phase, and their effect on vibration form B will depend on the numerical summation $\Sigma r \Delta_a \Delta_b$. But $\Sigma m \Delta_a \Delta_b = 0$, consequently there will be a coupling effect if the damping is not finely distributed with a proportional amount on each mass in the system.

11. Calculations in the actual case

Due to the symmetry a_{10s} is 0 in all vibration forms, and we have to do only with direct resonance of 8. order and secondary resonance of 6. order.

In vibration form A, fig. 5, we have $\beta_{aa} = 0.11$ and

$$a_{6s} = 6.25 \cdot 10^{-4} \qquad k = 0.094 \qquad b_8' = 0.59 \cdot 10^{-4}$$

$$\text{direct resonance:} \quad a_{8s} = 0.12 \cdot 10^{-4}$$

The correction member in (41 a) is here found as $a_{2u}^{*} = 0.001$, negligible in comparison with $\frac{1}{2}\beta \cdot \frac{6}{8} = 0.041$.

From vibration form B we find $\beta_{ba} = 0.09$ and

$$a_{6s} = 0.21 \cdot 10^{-4} \qquad d_1 = 0.26 \cdot 10^{-4} \qquad d_8' = 0.26 \cdot \frac{1}{2} \cdot 0.09 \cdot \frac{6}{8} = 0.01 \cdot 10^{-4}$$

From vibration form C: $\beta_{ca} = 0.15$

$$a_{6s} = 0.17 \cdot 10^{-4} \qquad e_1 = 0.19 \cdot 10^{-4} \qquad e_8' = 0.19 \cdot \frac{1}{2} \cdot 0.15 \cdot \frac{6}{8} = 0.01 \cdot 10^{-4}$$

Fig. 10 shows the principle of finding phase angles, when all impulse summations $\Sigma(\Delta)$ are made in the usual way for crank 1 in top. The vectors are rotating same way as the crankshaft, and the diagrams are then identical for both directions of revolution, except that the initial phase angle γ of the cylinder impulse (common for all cylinders) should be added in the direction of revolution.

The forces from vibration form B and C are inconsiderable in this case. Still the investigation of phase angles shows, that all 4 forces are nearly in phase together. The resultant is

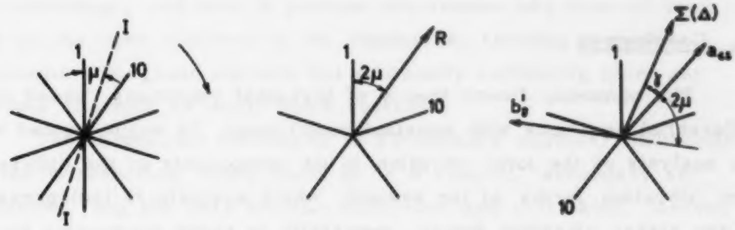


Fig. 10. Phase angle for fictive force.

$0.72 \cdot 10^{-4}$ corresponding to a static stress in the shaft at the node point $\tau_s = 3.80$ kg/cm^2. With a damping coefficient $\rho = 0.020$, usual for engines of this type, we finally get a vibration stress

$$\tau = F \cdot \tau_s \qquad \text{maximum } \frac{\tau_s}{\rho} = 190 \text{ kg/cm}^2$$

in close agreement with the measurements both at and about the resonance point.

12. Conclusions

The common, linear theory of torsional vibrations, based on differential equations with constant coefficients, is supplemented with the analysis of the total vibration in its components of the different free vibration forms of the system, which analysis is indispensable in any higher vibration theory, especially in cases concerning non-linear vibrations in multi-mass systems. As this analysis does not seem to be known by the authors of handbooks on vibration theory, it was found necessary here first to give a short description of it and its terminology.

The "second order", non-linear theory of crankshaft vibrations, based on differential equations with variable coefficients, comprises the effect of mass variation due to the oscillating masses, and of impulse distortion due to the vibrations.

Two interesting phenomena are caused by these effects which could not be explained by the simple, linear theory, namely sub-harmonic vibrations excited by impulses or mass variation components of frequency double of that of the vibration, and "secondary resonance" by which resonance vibrations are excited by impulses in resonance with a secondary component of the free vibration, of frequency different from the main vibration. Such secondary components (in the same vibration form) are caused by the mass variation.

These vibration phenomena are here calculated by means of "artificial impulses" and "fictive forces" representing the two above mentioned effects, whereby the crankshaft system can be regarded as a linear system on which the fictive forces act as additional outer impulses. The method is a "rationalization" of the mathematical method of iteration and gives a great simplification of detail calculations, especially in multi-mass systems. There is a close connection between the before mentioned vibration analysis and the application of fictive forces, because these forces are produced by, and acting upon, the "pure" vibration components in the

various free vibration forms, and each of these can be calculated exactly like one-mass systems. It is only by combination of the two matters (analysis and fictive forces) that they come to their full advantage, and this is perhaps the reason why none of them has so far been admitted to the handbooks, treating one-mass problems with great subtlety but cautiously refraining from any serious attempts on multi-mass systems.

The unexpected vibrations in 10-cylinder engines, mentioned in the introduction, were found to be a case of "secondary resonance", and the case is fully described and calculated, thereby also showing the smallness of certain corrections which may be disregarded in practical calculations.

Secondary resonance is a phenomenon common to all kinds of non-linear systems, and the calculation method here advanced may be useful in other cases. The principle is, that the impulse in secondary resonance will in the first place produce ordinary forced vibrations in the system, and these forced vibrations will then, through the special mechanism of non-linearity in the system considered, produce the exciting force for the main vibration. Secondary resonance in one vibration form may also be provoked by forced vibrations in other vibration forms of the system. The effect is fully investigated but found very small in the present case.

Subharmonic vibrations are also peculiar to all kinds of non-linear systems and due to similar effects as the secondary resonance - therefore the investigation of the two phenomena cannot be separated - but there is in our case a great difference between them. Secondary resonance in crankshaft systems is working exactly like ordinary resonance, only caused by an impulse which would not be expected in the linear theory, but the subharmonic vibrations are of a very different nature, arising sharply and unlimited by damping, if the damping is below a certain treshold value. The exciting fictive force is here produced by the vibration itself, not by a constant forced vibration, and the exciting force is therefore increasing with the vibration. In the curious case of "coupled subharmonic vibrations" free vibrations in two different vibration forms will each produce fictive forces exciting the other vibration form.

In crankshaft systems there will normally be damping enough to prevent subharmonic vibrations, or it will in any case be easy to provide for it, but secondary resonance must be taken as it is and included in the surveyance of all the other factors determining crank position and shaft dimensions.

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